Optimal decay of extremals for the fractional Sobolev inequality

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OPTIMAL DECAY OF EXTREMALS FOR THE FRACTIONAL SOBOLEV INEQUALITY

LORENZO BRASCO, SUNRA MOSCONI, AND MARCO SQUASSINA

Abstract. We obtain the sharp asymptotic behavior at infinity of extremal functions for the fractional critical Sobolev embedding.

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1. Introduction and main result

Let $N > p > 1$. In two seminal papers, T. Aubin [3] and G. Talenti [29] showed that the minimizers of the Sobolev quotient

$$\mathcal{S}_p = \inf_{u\in D^{1,p}(\mathbb{R}^N)} \left\{ \|\nabla u\|^p_{L^p(\mathbb{R}^N)} \,:\, \int_{\mathbb{R}^N} |u|^{Np/(N-p)}\,dx = 1 \right\}, \tag{1.1}$$

are given by the family of functions

$$U_t(x) = \mathcal{C}\, t^{\frac{p-N}{p}}\, U\left(\frac{x-x_0}{t}\right), \qquad \mathcal{C}\in\mathbb{R}\setminus\{0\},\ t>0,\ x_0\in\mathbb{R}^N, \tag{1.2}$$

where

$$U(x) = \mathcal{C}_{N,p}\left(1+|x|^{\frac{p}{p-1}}\right)^{\frac{p-N}{p}}, \qquad \mathcal{C}_{N,p} = \left(\int_{\mathbb{R}^N}\left(1+|x|^{\frac{p}{p-1}}\right)^{-N} dx\right)^{\frac{N\,p}{p-N}}.$$

For the limit case $p = 1$, the problem was investigated by H. Federer and W. H. Fleming in [13] and by V. G. Mazya in [24].

On one side, these results establish an enlightening connection between the theory of Sobolev spaces and the theory of classical isoperimetric inequalities. On the other side, they provide a very powerful tool for the study of second order partial differential equations

2010 *Mathematics Subject Classification.* 46E35, 35B40, 49K22.

Key words and phrases. Fractional $p-$Laplacian, critical Sobolev embedding, extremals, decay.

involving nonlinearities reaching the critical growth with respect to the Sobolev embedding. In the case $p=2$, these classification results were formally derived by G. Rosen in [27].

The variational problem (1.1) is related to the following equation involving the $p-$Laplace operator $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$,

$$-\Delta_p u = |u|^{\frac{N\,p}{N-p}-2}\,u, \qquad \text{in } \mathbb{R}^N. \tag{1.3}$$

In fact, a nontrivial problem is that of proving that the only *fixed sign* solutions of this equation are precisely given by (1.2).

In the restricted class of *radially symmetric fixed sign* solutions to (1.3), this was shown by M. Guedda and L. Veron in [16]. Recently, in [30, Corollary 1.3] for the case $1 < p \le 2\,N/(N+2)$, in [10, Theorem 1.2] for the case $2\,N/(N+2) < p \le 2$ and in [28, Theorem 1.1] for the case $2 < p < N$, it was proved that any positive weak solution to (1.3) is radially symmetric and radially decreasing about some point, thus answering positively to the classification of constant sign solutions to (1.3).

The result by Aubin and Talenti, as well as previous results in the linear case $p=2$, strongly rely on the reduction of the problem to the study of the radial solutions to an *ordinary differential equation* which can be explicitly solved. More recently, the Aubin-Talenti result has been reproved (and generalized) in [8, Theorem 2] by means of very different techniques, based on Optimal Transport.

Let now $s \in (0,1)$, $p > 1$ and $N > s\,p$. The goal of this paper is to provide information about the asymptotic behavior at infinity of optimizers of the problem

$$\mathcal{S}_{p,s} := \inf_{u\in D^{s,p}(\mathbb{R}^N)} \left\{ \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+s\,p}}\,dx\,dy \,:\, \int_{\mathbb{R}^N} |u|^{N\,p/(N-s\,p)}\,dx = 1 \right\}, \tag{1.4}$$

which is related to the fractional Sobolev embedding, see for example [25, Theorem 1]. Here

$$D^{s,p}(\mathbb{R}^N) = \left\{ u \in L^{N\,p/(N-s\,p)}(\mathbb{R}^N) : \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+s\,p}}\,dx\,dy < \infty \right\}.$$

In the limit case $p=1$, the sharp constant above has been determined in [14, Theorem 4.1] (see also [6, Theorem 4.10]). The relevant extremals are given by characteristic functions of balls, exactly as in the local case.

Problem (1.4) for $p>1$ is now related with the study of the nonlocal problem

$$(-\Delta_p)^s u = |u|^{\frac{N\,p}{N-s\,p}-2}\,u, \qquad \text{in } \mathbb{R}^N,$$

where, formally, the operator $(-\Delta_p)^s$ is defined on smooth functions as

$$(-\Delta_p)^s u(x) = 2\,\lim_{\varepsilon\searrow 0} \int_{\mathbb{R}^N\setminus B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p-2}\,(u(x)-u(y))}{|x-y|^{N+s\,p}}\,dy, \quad x\in\mathbb{R}^N,$$

up to a suitable normalization constant. This operator appears in some recent works like [2] and [19]. See also [11, 17, 18, 20] and the references therein for some existence and regularity results.

In the linear case $p=2$, it is known by [9] that the family of functions

$$U_t(x) = \mathcal{C}\,t^{\frac{2\,s-N}{2}} \left(1 + \left(\frac{|x-x_0|}{t}\right)^2\right)^{\frac{2\,s-N}{2}} \qquad \mathcal{C}\in\mathbb{R}\setminus\{0\},\ t>0,\ x_0\in\mathbb{R}^N,$$

is the only set of minimizers for the best Sobolev constant $\mathcal{S}_{2,s}$. It is also known [7] that, for a suitable positive constant $\mathcal{C}=\mathcal{C}(N,s)$, these are the only positive solutions of

$$(-\Delta)^s u = |u|^{\frac{N}{N-2\,s}-2}\,u \qquad \text{in } \mathbb{R}^N. \tag{1.5}$$

The result in [7] is based upon the full equivalence between the weak solutions to (1.5) and the integral formulation

$$u(x)=\int_{\mathbb{R}^N}\frac{u(y)^{\frac{N+2\,s}{N-2\,s}}}{|x-y|^{N-2\,s}}\,dy,\qquad u\in L^{\frac{2\,N}{N-2\,s}}(\mathbb{R}^N), \tag{1.6}$$

on the validity of some Kelvin transform and on moving plane arguments applied to (1.6), in the spirit of [21].

Unfortunately, in the nonlocal and nonlinear case $p\neq 2$ there is *no* Kelvin transform and *no* equivalent integral representation result. Furthermore, even restricting to the class of radially symmetric functions, establishing a classification result for the optimizers of (1.4) seems very hard. We conjecture that these are given by

$$U_t(x)=\mathcal{C}\,t^{\frac{s\,p-N}{p}}\,U\left(\frac{x-x_0}{t}\right),\qquad \mathcal{C}\in\mathbb{R}\setminus\{0\},\ t>0,\ x_0\in\mathbb{R}^N, \tag{1.7}$$

where this time

$$U(x):=\mathcal{C}_{N,p,s}\left(1+|x|^{\frac{p}{p-1}}\right)^{\frac{s\,p-N}{p}},\qquad \mathcal{C}_{N,p,s}=\left(\int_{\mathbb{R}^N}\left(1+|x|^{\frac{p}{p-1}}\right)^{-N}dx\right)^{\frac{N\,p}{s\,p-N}}. \tag{1.8}$$

Notice that (1.7) and (1.8) are consistent with the case $p=2$ or $s=1$, in the last case we are back to the family of Aubin-Talenti functions (1.2) for the $p-$Laplacian operator.

In the main result of this paper, we prove that extremals for (1.4) have exactly the decay rate at infinity dictated by formula (1.8). Namely, we have the following.

Theorem 1.1. *Let $U\in D^{s,p}(\mathbb{R}^N)$ be any solution to (1.4). Then $U\in L^\infty(\mathbb{R}^N)$ is a constant sign, radially symmetric and monotone function with*

$$\lim_{|x|\to\infty}|x|^{\frac{N-s\,p}{p-1}}\,U(x)=C, \tag{1.9}$$

for some constant $C\in\mathbb{R}\setminus\{0\}$.

The building blocks of Theorem 1.1 are a weak L^q estimate for the minimizers (Proposition 3.3), a Radial Lemma for Lorentz spaces (Lemma 2.8) and the fact that the function

$$\Gamma(x):=|x|^{-\frac{N-s\,p}{p-1}},\quad x\in\mathbb{R}^N\setminus\{0\},$$

is a weak solution of $(-\Delta_p)^s u=0$ in $\mathbb{R}^N\setminus B_r$, for any $r>0$ (Theorem A.4). Then the crucial point will be constructing suitable barrier functions to be combined with a version of the comparison principle for $(-\Delta_p)^s$ recently obtained in [18]. Observe that for $s=1$, the function Γ above is nothing but the fundamental solution of the $p-$Laplacian.

We wish to stress that Theorem 1.1 also provides a very useful tool for the investigation of *existence* of weak solutions for the nonlocal Brezis-Nirenberg problem in a smooth bounded domain Ω, i.e.

$$\begin{cases}(-\Delta_p)^s\,u=\lambda\,|u|^{p-2}\,u+|u|^{\frac{N\,p-2\,(N-s\,p)}{N-s\,p}}\,u & \text{in } \Omega,\\ u=0 & \text{in } \mathbb{R}^N\setminus\Omega,\end{cases}$$

where $\lambda > 0$. In fact, by means of (1.9), one can estimate truncations of U_t via a suitable cut-off function in terms of the sharp constant $\mathcal{S}_{p,s}$ *without knowing the explicit form* of the optimizers. Such a procedure is new even for the local case. These estimates allow to apply mountain pass or linking arguments by forcing the min-max levels to fall inside a compactness range for the energy functionals, see [26] for more details.

Plan of the paper. In Section 2 we set all the notations, definitions and basic facts that will be needed throughout the paper. Then in Section 3 we prove existence of solutions for (1.4), together with some basic integrability properties. We also prove that extremals have to comparable to

$$x \mapsto |x|^{-\frac{N-s\,p}{p-1}}, \quad x \in \mathbb{R}^N \setminus \{0\},$$

at infinity (Corollary 3.7). Then the exact behavior (1.9) is proved in Section 4. The paper ends with Appendix A, containing a rigourous computation of the fractional $p-$Laplacian of a power function.

Acknowledgements. We warmly thank Yannick Sire for some informal discussions on the subject of this paper. This research has been partially supported by *Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni* (INdAM) and by *Agence Nationale de la Recherche*, through the project ANR-12-BS01-0014-01 Geometrya. Part of this paper was written during a visit of S. M. and M. S. in Marseille in March 2015. The I2M and FRUMAM institutions are gratefully acknowledged.

2. Preliminary results

2.1. **Notation.** In the following we will fix $s \in (0,1)$, $p > 1$ and N as the dimension, letting for brevity

$$p^* = \frac{N\,p}{N-s\,p}.$$

Moreover, $\mathbf{S}^{N-1}$ will denote $\{x \in \mathbb{R}^N : |x| = 1\}$. For $E \subseteq \mathbb{R}^N$ measurable we denote by $|E|$ is Lebesgue measure, let $E^c = \mathbb{R}^N \setminus E$ with χ_E its characteristic function. If $u : E \to \mathbb{R}$ is measurable we set

$$[u]^p_{W^{s,p}(E)} := \int_{E\times E} \frac{|u(x)-u(y)|^p}{|x-y|^{N+s\,p}}\,dx\,dy, \qquad [u]_{s,p} = [u]_{W^{s,p}(\mathbb{R}^N)},$$

and for any $q > 1$

$$\|u\|_q := \left(\int_{\mathbb{R}^N} |u|^q\,dx\right)^{1/q}.$$

2.2. **Elementary inequalities.** For $t \in \mathbb{R}$ we define

$$J_p(t) = |t|^{p-2}\,t.$$

We will now consider some useful inequalities on the function J_p. First, consider the case $p \geq 2$. We recall that

$$|J_p(a) - J_p(b)| \leq (p-1)\,(|a|^{p-2} + |b|^{p-2})\,|a-b|, \qquad a,b \in \mathbb{R}, \quad p \geq 2, \tag{2.1}$$

as a consequence of the mean value Theorem. Notice also that J_p is convex on $[0, +\infty[$ in this case, and the following consequences of convexity hold true

$$|J_p(a+b)| \leq C_p(|a|^{p-1} + |b|^{p-1}), \qquad a,b \in \mathbb{R}, \quad p \geq 2.$$

In [18, eq. (2.7)] it is also proved the following inequality

$$J_p(a) - J_p(a+b) \le -2^{2-p}\, b^{p-1}, \qquad a \in \mathbb{R},\ b \ge 0, \quad p \ge 2. \tag{2.2}$$

Next we consider the case $p \in [1,2]$. The well known subadditivity inequality reads

$$|J_p(a+b)| \le |a|^{p-1} + |b|^{p-1}, \qquad a, b \in \mathbb{R}, \quad p \in [1,2],$$

or

$$J_p(a+b) \le J_p(a) + J_p(a), \qquad a, b \ge 0, \quad p \in [1,2].$$

We also recall the well-known monotonicity inequality

$$\big(J_p(a) - J_p(b)\big)\,(a-b) \ge c\,\frac{|a-b|^2}{(a^2+b^2)^{\frac{2-p}{2}}}, \qquad a, b \in \mathbb{R}\setminus\{0\}, \quad p \in [1,2]. \tag{2.3}$$

Next we prove the following inequality

$$J_p(a) - J_p(a-b) \ge \max\left\{ J_p(A) - J_p(A-b),\ \left(\frac{b}{2}\right)^{p-1} \right\}, \quad a \in [0,A], b \ge 0,\ p \in [1,2]. \tag{2.4}$$

We distinguish two cases. First suppose that $a \ge b/2$. The function $t \mapsto J_p(t) - J_p(t-b)$ is readily seen to be decreasing on $[b/2, +\infty[$, so that

$$J_p(a) - J_p(a-b) \ge J_p(A) - J_p(A-b)$$

in this case. On the other hand if $a < b/2$ then since J_p is odd and increasing we have

$$J_p(a) - J_p(a-b) \ge J_p(b-a) \ge J_p\left(\frac{b}{2}\right),$$

and thus (2.4) is proved.

2.3. **Functional framework.**

We consider the space

$$D_0^{s,p}(\Omega) := \Big\{ u \in L^{p^*}(\Omega) : u \equiv 0 \text{ in } \Omega^c,\ [u]_{s,p} < +\infty \Big\}, \qquad D_0^{s,p}(\mathbb{R}^N) := D^{s,p}(\mathbb{R}^N),$$

which is a Banach space with respect to the norm $[\,\cdot\,]_{s,p}$. Our first aim is to prove, under suitable regularity assumptions on $\partial\Omega$, that $C_c^\infty(\Omega)$ is dense in $D_0^{s,p}(\Omega)$ with respect to the norm $[\,\cdot\,]_{s,p}$. While this density result is well-known for bounded regular domains (see for example [12]), we will need to consider exterior unbounded domains in the following. Finally we will prove a comparison principle in a rather general space.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^N$ be an open set such that $\partial\Omega$ is compact and locally the graph of a continuous function. Then $D_0^{s,p}(\Omega)$ is the completion of $C_c^\infty(\Omega)$ with respect to the norm $[\,\cdot\,]_{s,p}$.*

Proof. Let $u \in D_0^{s,p}(\Omega)$. Reasoning on u_+ and u_- separately (which still belong to $D_0^{s,p}(\Omega)$), we can suppose that u is nonnegative. Consider, for $\varepsilon > 0$, the function $u_\varepsilon = (u-\varepsilon)_+$. Using the $1-$Lipschitzianity of $t \mapsto (t-\varepsilon)_+$ it is readily checked that

$$|u_\varepsilon(x) - u_\varepsilon(y)|^p \le |u(x) - u(y)|^p, \qquad |u_\varepsilon(x) - u_\varepsilon(y)|^p \to |u(x) - u(y)|^p, \quad \text{a.e. in } \mathbb{R}^{2N}.$$

Therefore $u_\varepsilon \in D_0^{s,p}(\Omega)$ and by dominated convergence $[u_\varepsilon]_{s,p} \to [u]_{s,p}$. This in turn implies that $u_\varepsilon \to u$ in $D_0^{s,p}(\Omega)$ by uniform convexity of the norm. Now Chebyshev's inequality ensures that $\mathrm{supp}(u_\varepsilon)$ has finite measure, thus by Hölder's inequality we get $u_\varepsilon \in L^p(\mathbb{R}^N)$. This yields

$$u_\varepsilon \in X_0^{s,p}(\Omega) := D_0^{s,p}(\Omega) \cap L^p(\mathbb{R}^N),$$

and [12, Theorem 6] ensures that u_ε can be approximated, in the norm $[\,\cdot\,]_{s,p}$, by functions which belong to $C^\infty_c(\Omega)$. □

We recall the following nonlocal Hardy inequality proved in [14, Theorem 2].

Proposition 2.2 (Hardy's inequality)**.** *Let $N > s\,p$. Then there exists $C = C(N, p, s) > 0$ such that*

$$\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{s\,p}}\,dx \le C\,[u]^p_{s,p}, \qquad \textit{for every } u \in D^{s,p}(\mathbb{R}^N). \tag{2.5}$$

We then define a suitable space where a comparison principle holds true. For any $\Omega \subset \mathbb{R}^N$ open set, we define

$$\widetilde{D}^{s,p}(\Omega) := \Big\{ u \in L^{p-1}_{\rm loc}(\mathbb{R}^N) \cap L^{p^*}(\Omega) \,:\exists\, E \supset \Omega \text{ with } E^c \text{ compact, } \mathrm{dist}(E^c, \Omega) > 0 \\ \text{and } [u]_{W^{s,p}(E)} < +\infty \Big\}.$$

We wish to point out that the definition above is given having in mind the case of Ω being an exterior domain, i.e. the complement of a compact set. Essentially, we consider functions u which are regular in a slight enlargement of Ω and possibly rough far from Ω.

Lemma 2.3. *Let $1 < p < \infty$ and $0 < s < 1$. For every $u \in L^{p-1}_{\rm loc}(\mathbb{R}^N)$, every $E \subset \mathbb{R}^N$ open set and every ball $B_R \subset E$, we have*

$$\int_E \frac{|u(x)|^p}{(1 + |x|)^{N+s\,p}}\,dx \le C\,[u]^p_{W^{s,p}(E)} + C \left(\int_{B_R} |u|^{p-1}\,dx \right)^{\frac{p}{p-1}}, \tag{2.6}$$

for some $C = C(N, p, s, R) > 0$, blowing-up as $R \to 0$.

Proof. We assume that the right-hand side on (2.6) is finite, otherwise there is nothing to prove. For simplicity, we can suppose that B_R is centered at the origin. Then by a standard compactness argument, we can obtain the interpolation inequality

$$\int_{B_R} |u|^p\,dx \le \frac{C}{R^{s\,p}}\,[u]^p_{W^{s,p}(B_R)} + \frac{C}{R^{\frac{N}{p-1}}} \left(\int_{B_R} |u|^{p-1}\,dx \right)^{\frac{p}{p-1}}, \tag{2.7}$$

for some constant $C = C(N, p, s) > 0$. In particular, we also have

$$\int_{B_R} \frac{|u|^p}{(1 + |x|)^{N+s\,p}}\,dx \le \frac{C}{R^{s\,p}}\,[u]^p_{W^{s,p}(B_R)} + \frac{C}{R^{\frac{N}{p-1}}} \left(\int_{B_R} |u|^{p-1}\,dx \right)^{\frac{p}{p-1}}. \tag{2.8}$$

We then take the smaller ball $B_{R/2}$ (still centered at the origin), we have

$$\int_{E \setminus B_R} \int_{B_{R/2}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+s\,p}}\,dy\,dx \le [u]^p_{W^{s,p}(E)} < +\infty.$$

There holds

$$|x - y| \le \frac{3}{2}\,|x|, \qquad x \in E \setminus B_R,\ y \in B_{R/2},$$

thus we get

$$\int_{E\setminus B_R}\int_{B_{R/2}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+s\,p}}\,dy\,dx \geq c\,R^N \int_{E\setminus B_R} \frac{|u|^p}{|x|^{N+s\,p}}\,dx - c\left(\int_{E\setminus B_R} \frac{1}{|x|^{N+s\,p}}\,dx\right)\int_{B_{R/2}} |u|^p\,dy.$$

In conclusion, the previous estimate proves

$$\int_{E\setminus B_R} \frac{|u|^p}{(1+|x|)^{N+s\,p}}\,dx \leq \frac{C}{R^N}\,[u]^p_{W^{s,p}(E)} + \frac{C}{R^{s\,p}}\int_{B_{R/2}} |u|^p\,dy.$$

We use (2.7) in the right-hand side, this gives

$$\int_{E\setminus B_R} \frac{|u|^p}{(1+|x|)^{N+s\,p}}\,dx \leq C_R\,[u]^p_{W^{s,p}(E)} + C_R\left(\int_{B_{R/2}} |u|^{p-1}\,dy\right)^{\frac{p}{p-1}}. \tag{2.9}$$

By summing up (2.8) and (2.9) we get the conclusion. $\square$

The next proposition shows that in the space $\widetilde{D}^{s,p}(\Omega)$, the operator $(-\Delta_p)^s$ is well defined.

Proposition 2.4. *For any $u \in \widetilde{D}^{s,p}(\Omega)$, the operator*

$$D^{s,p}_0(\Omega) \ni \varphi \mapsto \langle(-\Delta_p)^s u, \varphi\rangle := \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{J_p(u(x)-u(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{N+s\,p}}\,dx\,dy$$

is well defined and belongs to the dual space $(D^{s,p}_0(\Omega))^*$.

Proof. We proceed as in [18, Lemma 2.3]. Let $E \supset \Omega$ be such that E^c is compact, $\mathrm{dist}(E^c,\Omega) > 0$ and $[u]_{W^{s,p}(E)} < +\infty$. Since $\varphi \equiv 0$ in Ω^c, we split the integral as

$$\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{J_p(u(x)-u(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dx\,dy = \int_{E\times E} \frac{J_p(u(x)-u(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}} + 2\int_{\Omega\times E^c} \frac{J_p(u(x)-u(y))\,\varphi(x)}{|x-y|^{N+sp}}\,dx\,dy.$$

By Hölder's inequality the first term is finite and defines a continuous linear functional on $D^{s,p}_0(\Omega)$. Let us focus on the second one. By using that $\varphi \equiv 0$ in E^c, we need to show that

$$\varphi \mapsto \int_\Omega \varphi(x)\left(\int_{E^c} \frac{J_p(u(x)-u(y))}{|x-y|^{N+sp}}\,dy\right)dx,$$

is a continuous linear functional on $D^{s,p}_0(\Omega)$. By means of Hardy's inequality (2.5), we get that convergence of $\{\varphi_n\}_{n\in\mathbb{N}}$ in $D^{s,p}_0(\Omega) \subset D^{s,p}(\mathbb{R}^N)$ implies strong convergence in $L^p(\Omega)$ of $\{|x|^{-s}\varphi_n\}_{n\in\mathbb{N}}$. Thus to prove the claim it suffices to show that

$$x \mapsto |x|^s \int_{E^c} \frac{J_p(u(x)-u(y))}{|x-y|^{N+s\,p}}\,dy \in L^{p'}(\Omega).$$

Being E^c compact and $\mathrm{dist}(E^c,\Omega) \geq \delta > 0$ it holds

$$|x-y| \geq C\,(1+|x|), \qquad \text{for every } x \in \Omega,\ y \in E^c, \tag{2.10}$$

for some $C = C(E, \Omega) > 0$. Thus, for almost every $x \in \Omega$, we can estimate

$$\left|\int_{E^c} \frac{|x|^s\, J_p(u(x)-u(y))}{|x-y|^{N+sp}}\,dy\right| \le C\left[|E^c|\,\frac{|u(x)|^{p-1}}{(1+|x|)^{\frac{N+sp}{p'}}} + \frac{1}{(1+|x|)^{N+s\,(p-1)}}\int_{E^c}|u|^{p-1}\,dy\right].$$

The first term belongs to $L^{p'}(\Omega)$ due to (2.6). For the second one this follows from a direct computation. This proves the claim and the proposition. □

Definition 2.5. Let $u \in \widetilde{D}^{s,p}(\Omega)$ and $\Lambda \in (D_0^{s,p}(\Omega))^*$. We say that $(-\Delta_p)^s u \le \Lambda$ weakly in Ω if for all $\varphi \in D_0^{s,p}(\Omega)$, $\varphi \ge 0$ in Ω,

$$\int_{\mathbb{R}^{2N}} \frac{J_p(u(x)-u(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{N+s\,p}}\,dx\,dy \le \langle \Lambda, \varphi\rangle.$$

Theorem 2.6 (Comparison principle in general domains)**.** *Let $\Omega \subset \mathbb{R}^N$ be an open set. Let $u, v \in \widetilde{D}^{s,p}(\Omega)$ satisfy*

$$u \le v, \quad in\ \Omega^c \qquad and \qquad (-\Delta_p)^s u \le (-\Delta_p)^s v, \quad in\ \Omega.$$

Then $u \le v$ in Ω.

Proof. It suffices to proceed as in [23, Lemma 9], we only need to prove that $w := (u-v)_+ \in D_0^{s,p}(\Omega)$ is an admissible test function. Clearly $w \equiv 0$ in Ω^c and $w \in L^{p^*}(\mathbb{R}^N)$. To estimate the Gagliardo seminorm, let $E \supset \Omega$ be such that E^c is compact, $\mathrm{dist}(E^c, \Omega) > 0$ and

$$\int_{E\times E} \frac{|u(x)-u(y)|^p}{|x-y|^{N+s\,p}}\,dx\,dy + \int_{E\times E} \frac{|v(x)-v(y)|^p}{|x-y|^{N+s\,p}}\,dx\,dy < +\infty. \tag{2.11}$$

Then

$$\int_{\mathbb{R}^{2\,N}} \frac{|w(x)-w(y)|^p}{|x-y|^{N+s\,p}}\,dx\,dy = \int_{E\times E} \frac{|w(x)-w(y)|^p}{|x-y|^{N+s\,p}}\,dx\,dy + 2\int_{\Omega\times E^c} \frac{|w(x)|^p}{|x-y|^{N+s\,p}}\,dx\,dy,$$

and the first integral is finite due to

$$|w(x)-w(y)|^p \le 2^p\,(|u(x)-u(y)|^p + |v(x)-v(y)|^p)$$

and (2.11). For the second one we use (2.10), and since $|w(x)|^p \le C(|u(x)|^p + |v(x)|^p)$ we get

$$\int_{\Omega\times E^c} \frac{|w(x)|^p}{|x-y|^{N+sp}}\,dx\,dy \le C\,|E^c|\int_\Omega \frac{|u(x)|^p}{(1+|x|)^{N+sp}}\,dx + C\,|E^c|\int_\Omega \frac{|v(x)|^p}{(1+|x|)^{N+sp}}\,dx.$$

The last two terms are finite, due the definition of $\widetilde{D}^{s,p}(\Omega)$ and (2.6). □

Finally, for the reader's convenience we recall the following result from [18]. The proof is identical to the one of [18, Lemma 2.8] and we omit it.

Proposition 2.7 (Non-local behavior of $(-\Delta_p)^s$)**.** *Let $N > s\,p$ and let $\Omega \subset \mathbb{R}^N$ be an open set such that $\partial\Omega$ is compact and locally the graph of continuous functions. Suppose that $u \in \widetilde{D}^{s,p}(\Omega)$ solves $(-\Delta_p)^s u = f$ for some $f \in L^1_{\mathrm{loc}}(\Omega) \cap (D_0^{s,p}(\Omega))^*$, in the sense that*

$$\langle (-\Delta_p)^s u, \varphi\rangle = \int_\Omega f\,\varphi\,dx, \qquad for\ every\ \varphi \in D_0^{s,p}(\Omega). \tag{2.12}$$

Let v be a measurable function with compact support $K := \mathrm{supp}(v)$ such that

$$\mathrm{dist}(K, \Omega) > 0, \qquad \int_{\Omega^c} |v|^{p-1}\,dx < +\infty,$$

and define for a.e. Lebesgue point $x \in \Omega$ of u

$$h(x) = 2 \int_K \frac{J_p\big(\big(u(x)-u(y)\big)-v(y)\big) - J_p\big(u(x)-u(y)\big)}{|x-y|^{N+s\,p}}\, dy.$$

Then $u+v \in \widetilde{D}^{s,p}(\Omega)$ and $(-\Delta_p)^s(u+v) = f+h$ in weak sense.

2.4. **Radial functions.** For every measurable function $u : \mathbb{R}^N \to \mathbb{R}$ we define its distribution function

$$\mu_u(t) = \big|\{x\,:\,|u(x)|>t\}\big|, \qquad t>0.$$

Let $1 \le q < \infty$ and $1 \le \theta < \infty$, the *Lorentz space* $L^{q,\theta}(\mathbb{R}^N)$ is defined by

$$L^{q,\theta}(\mathbb{R}^N) = \left\{ u\,:\, \int_0^\infty t^{\theta-1}\,\mu_u(t)^{\frac{\theta}{q}}\,dt < +\infty \right\}.$$

In the limit case $\theta=\infty$, this is defined by

$$L^{q,\infty}(\mathbb{R}^N) = \left\{ u\,:\, \sup_{t>0} t\,\mu(t)^{\frac{1}{q}} < +\infty \right\},$$

and we recall that this coincides with the weak L^q space.

Lemma 2.8 (Radial Lemma for Lorentz spaces)**.** *Let $1 \le \theta \le \infty$ and $1 \le q < \infty$. Let $u \in L^{q,\theta}(\mathbb{R}^N)$ be a positive and radially symmetric decreasing function. Then the following decay estimates holds true:*

$$0 \le u(x) \le \left(\frac{\theta\,\omega_N^{-\frac{\theta}{q}}}{N} \int_0^\infty t^{\theta-1}\,\mu(t)^{\frac{\theta}{q}}\,dt \right)^{\frac{1}{\theta}} |x|^{-\frac{N}{q}}, \qquad \text{if } \theta<\infty,$$

and

$$0 \le u(x) \le \left(\omega_N^{-\frac{1}{q}} \sup_{t>0} t\,\mu(t)^{\frac{1}{q}} \right) |x|^{-\frac{N}{q}}, \qquad \text{if } \theta=\infty.$$

Proof. We start with the case $\theta<\infty$. First of all, we prove that

$$\int_0^\infty t^{\theta-1}\,\mu(t)^{\frac{\theta}{q}}\,dt = \frac{N-\alpha}{\theta\,\omega_N^{\alpha/N}} \int_{\mathbb{R}^N} \frac{|u|^\theta}{|x|^\alpha}\,dx, \tag{2.13}$$

where the exponent $\alpha<N$ given by the relation[1]

$$\frac{\theta}{q} = \frac{N-\alpha}{N}.$$

With a simple change of variable

$$\int_0^\infty t^{\theta-1}\,\mu(t)^{\frac{\theta}{q}}\,dt = \frac{1}{\theta}\int_0^\infty \mu(s^{1/\theta})^{\frac{\theta}{q}}\,ds. \tag{2.14}$$

Then we observe that

$$\int_{\mathbb{R}^N} \frac{|u|^\theta}{|x|^\alpha}\,dx = \int_{\mathbb{R}^N} \frac{\displaystyle\int_0^\infty \chi_{\{u(x)^\theta>t\}}(s)\,ds}{|x|^\alpha}\,dx = \int_0^\infty \int_{\mathbb{R}^N} \frac{\chi_{\{u(x)>t^{1/\theta}\}}(s)}{|x|^\alpha}\,dx\,ds,$$

[1] Observe that if $\theta>q$, then $\alpha<0$.

and by assumption we have

$$\{x\,:\,u(x)>t^{1/\theta}\}=\left\{x\,:\,|x|<\left(\frac{\mu(t^{1/\theta})}{\omega_N}\right)^{\frac{1}{N}}\right\}=:B_{R(t)},$$

since the function u is radially decreasing. Thus we arrive at

$$\begin{aligned}\int_{\mathbb{R}^N}\frac{|u|^\theta}{|x|^\alpha}\,dx&=\int_0^\infty\left(\int_{B_{R(s)}}\frac{1}{|x|^\alpha}\,dx\right)ds\\&=\omega_N\int_0^\infty\int_0^{R(s)}\varrho^{N-1-\alpha}\,d\varrho\,ds=\frac{\omega_N}{N-\alpha}\int_0^\infty\frac{\mu(s^{1/\theta})^{\frac{N-\alpha}{N}}}{\omega_N^{\frac{N-\alpha}{N}}}\,ds\\&=\frac{\omega_N^{\alpha/N}}{N-\alpha}\int_0^\infty\mu(s^{1/\theta})^{\frac{\theta}{q}}\,ds\end{aligned}$$

By using (2.14), we finally obtain

$$\int_0^\infty t^{\theta-1}\,\mu(t)^{\frac{\theta}{q}}\,dt=\frac{N-\alpha}{\theta\,\omega_N^{\alpha/N}}\int_{\mathbb{R}^N}\frac{|u|^\theta}{|x|^\alpha}\,dx,$$

which proves (2.13).

As for the decay estimate, thanks to (2.13) we have

$$\begin{aligned}+\infty>\int_{\mathbb{R}^N}\frac{|u|^\theta}{|x|^\alpha}\,dx&=N\,\omega_N\int_0^{+\infty}|u(\varrho)|^\theta\,\varrho^{N-1-\alpha}\,d\varrho\\&\geq N\,\omega_N\int_0^R u(\varrho)^\theta\,\varrho^{N-1-\alpha}\,d\varrho\geq N\,\omega_N\,u(R)^\theta\,\frac{R^{N-\alpha}}{N-\alpha},\end{aligned}$$

where α is as above. By recalling that $(N-\alpha)/\theta=N/q$, we get the desired conclusion.

For the case $\theta=\infty$, it is sufficient to observe that

$$\sup_{t>0}t\,\mu(t)^{\frac{1}{q}}=\omega_N^{\frac{1}{q}}\sup_{x\in\mathbb{R}^N}|x|^{\frac{N}{q}}\,u(x).$$

Then the decay estimate easily follows. □

3. Properties of extremals

3.1. Basic properties.

We start with the following result.

Proposition 3.1. *Let $1<p<\infty$ and $s\in(0,1)$ be such that $s\,p<N$. Then:*

- *the variational problem* (1.4) *admits a solution;*
- *for every $U\in D^{s,p}(\mathbb{R}^N)$ solving* (1.4)*, there exist $x_0\in\mathbb{R}^N$ and $u:\mathbb{R}^+\to\mathbb{R}$ constant sign monotone function such that $U(x)=u(|x-x_0|)$;*
- *every minimizer $U\in D^{s,p}(\mathbb{R}^N)$ weakly solves*

$$(-\Delta_p)^sU=\mathcal{S}_{p,s}\,|U|^{p^*-2}\,U,\qquad \text{in } \mathbb{R}^N,$$

that is

$$\int_{\mathbb{R}^{2N}} \frac{J_p(U(x)-U(y))\,\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{N+s\,p}}\,dx\,dy = \mathcal{S}_{p,s}\int_{\mathbb{R}^N} |U|^{p^*-2}\,U\,\varphi\,dx, \tag{3.1}$$

for every $\varphi \in D^{s,p}(\mathbb{R}^N)$.

Proof. Existence of a solution for (1.4) follows from the Concentration-Compactness Principle, see [22, Section I.4, Example iii)]. It is not difficult to show that every solution of (1.4) must have costant sign. Indeed, for every admissible $u \in D^{s,p}(\mathbb{R}^N)$, the function $|u|$ is still admissible and we have

$$\big||u(x)|-|u(y)|\big| \le |u(x)-u(y)|.$$

More important, the inequality sign is strict if $u(x)\,u(y)<0$, i.e. if u changes sign. Radiality of the solutions then comes from the *Pólya-Szegő principle* for Gagliardo seminorms (see [1]), i.e. for every positive function $u \in D^{s,p}(\mathbb{R}^N)$ we have

$$[u^\#]^p_{s,p} \le [u]^p_{s,p}. \tag{3.2}$$

Here $u^\#$ denotes the radially symmetric decreasing rearrangement of u. It is crucial to observe that inequality (3.2) is strict, unless u is (up to a translation) a radially symmetric decreasing function, see [14, Theorem A.1].

Finally, if U solves (1.4), then it minimizes as well the functional

$$u \mapsto [u]^p_{s,p} - \mathcal{S}_{p,s}\left(\int_{\mathbb{R}^N} |u|^{p^*}\,dx\right)^{\frac{p}{p^*}}.$$

Equation (3.1) is exactly the relevant Euler-Lagrange equation associated with this functional, once it is observed that U has unitary L^{p^*} norm and constant sign. □

Proposition 3.2 (Global boundedness)**.** *Let* $U \in D^{s,p}(\mathbb{R}^N)$ *be a positive solution of* (1.4)*. Then we have* $U \in L^\infty(\mathbb{R}^N)$.

Proof. Thanks to the properties of the minimizers contained in Proposition 3.1, it is enough to prove that $U \in L^\infty_{\rm loc}(\mathbb{R}^N)$. At this aim, we just need to show that $U \in L^{q\,(p^*-1)}(\mathbb{R}^N)$ for some $q > N/(s\,p)$. This would imply that

$$U^{p^*-1} \in L^q(\mathbb{R}^N), \qquad \text{for some } q > \frac{N}{s\,p},$$

and thus $U \in L^\infty_{\rm loc}(\mathbb{R}^N)$ would automatically follow by [5, Theorem 3.8].

Let $M>0$ and $\alpha>1$, we set for simplicity $U_M=\min\{U,M\}$ and $g_{\alpha,M}(t)=t\,\min\{t,\,M\}^{\alpha-1}$. Then we insert in (3.1) the test function $\varphi = g_{\alpha,M}(U) \in D^{s,p}(\mathbb{R}^N)$. This yields

$$\int_{\mathbb{R}^{2N}} \frac{J_p(U(x)-U(y))\,\big(g_{\alpha,M}(U(x))-g_{\alpha,M}(U(y))\big)}{|x-y|^{N+s\,p}}\,dx\,dy = \mathcal{S}_{p,s}\int_{\mathbb{R}^N} U^{p^*-p}\,U_M^{\alpha-1}\,U^p\,dx.$$

We now observe that if we set

$$G_{\alpha,M}(t)=\int_0^t g'_{\alpha,M}(\tau)^{\frac{1}{p}}\,d\tau,$$

by using [5, Lemma A.2] from the previous identity with simple manipulations we get

$$\int_{\mathbb{R}^{2N}} \frac{|G_{\alpha,M}(U(x)) - G_{\alpha,M}(U(y))|^p}{|x-y|^{N+s\,p}}\,dx\,dy$$
$$\le \mathcal{S}_{p,s}\left[K_0^{\alpha-1}\int_{\mathbb{R}^N} U^{p^*}\,dx + \left(\int_{\{U\ge K_0\}} U^{p^*}\,dx\right)^{\frac{p^*-p}{p^*}} \left(\int_{\mathbb{R}^N}\left(U_M^{(\alpha-1)}\,U^p\right)^{\frac{p^*}{p}}\,dx\right)^{\frac{p}{p^*}}\right],$$

for some $K_0>0$ that will be chosen in a while. If we estimate from below the left-hand side by Sobolev inequality, we get[2]

$$\left(\frac{p}{\alpha-1+p}\right)^p \left(\int_{\mathbb{R}^N}\left(U^p\,U_M^{(\alpha-1)}\right)^{\frac{p^*}{p}}\,dx\right)^{\frac{p}{p^*}} \le K_0^{\alpha-1} + \left(\int_{\{U\ge K_0\}} U^{p^*}\,dx\right)^{\frac{p^*-p}{p^*}} \left(\int_{\mathbb{R}^N}\left(U_M^{(\alpha-1)}\,U^p\right)^{\frac{p^*}{p}}\,dx\right)^{\frac{p}{p^*}}. \tag{3.3}$$

We now choose the parameters: we first take $\alpha>1$ such that

$$p^* + (\alpha-1)\,\frac{p^*}{p} = q\,(p^*-1), \qquad \text{i.e.}\quad \alpha = p\,q\,\frac{(p^*-1)}{p^*} - (p-1),$$

where $q>N/(s\,p)$, then we choose $K_0=K_0(\alpha,U)>0$ such that

$$\left(\int_{\{U\ge K_0\}} U^{p^*}\,dx\right)^{\frac{p^*-p}{p^*}} \le \frac{1}{2}\left(\frac{p^*}{q\,(p^*-1)}\right)^p.$$

With this choice we can absorb the last term on the right-hand side of (3.3) and thus obtain

$$\left(\frac{p^*}{q\,(p^*-1)}\right)^p \left(\int_{\mathbb{R}^N} U^{p^*}\,U_M^{(\alpha-1)\,\frac{p^*}{p}}\,dx\right)^{\frac{p}{p^*}} \le 2\,K_0^{\alpha-1}.$$

If we now take the limit as M goes to $+\infty$, we finally get that $U\in L^{q\,(p^*-1)}(\mathbb{R}^N)$ for some $q>N/(s\,p)$, together with the estimate

$$\left\|U^{p^*-1}\right\|_q^q \le \left(2\,K_0^{\alpha-1}\left(q\,\frac{p^*-1}{p^*}\right)^p\right)^{\frac{p^*}{p}},$$

and thus the conclusion. □

Proposition 3.3 (Borderline Lorentz estimate)**.** *Let $U\in D^{s,p}(\mathbb{R}^N)$ be a positive solution of (1.4). Then*

$$U\in L^q(\mathbb{R}^N), \qquad \text{for every } q>\vartheta := \frac{(p-1)\,N}{N-s\,p}. \tag{3.4}$$

[2]Here we use that

$$G_{\alpha,M}(t) \ge \frac{p}{p+\alpha-1}\,t\,\min\{t,\,M\}^{\frac{\alpha-1}{p}}.$$

Moreover, we have $U \in L^{\vartheta,\infty}(\mathbb{R}^N)$

$$\sup_{t>0} t\,|\{U>t\}|^{\frac{1}{\vartheta}} \le \|U\|_{p^*-1}^{\frac{p^*-1}{p-1}}. \tag{3.5}$$

Proof. We divide the proof in two parts: we first prove (3.4). Then we will use (3.4) to prove (3.5).

Part I: intermediate estimate. Given $0<\alpha<1$ and $\varepsilon>0$, we take the Lipschitz increasing function $g_\varepsilon:[0,+\infty)\to[0,+\infty)$ defined as

$$g_\varepsilon(t)=\int_0^t \left[(\varepsilon+\tau)^{\frac{\alpha-1}{p}}+\frac{\alpha-1}{p}\,\tau\,(\varepsilon+\tau)^{\frac{\alpha-1-p}{p}}\right]^p\,d\tau.$$

We insert in (3.1) the test function $\varphi=g_\varepsilon(U)\in D^{s,p}(\mathbb{R}^N)$. This gives

$$\int_{\mathbb{R}^{2N}} \frac{J_p(U(x)-U(y))\,\big(g_\varepsilon(U(x))-g_\varepsilon(U(y))\big)}{|x-y|^{N+s\,p}}\,dx\,dy=\mathcal{S}_{p,s}\int_{\mathbb{R}^N} U^{p^*-1}\,g_\varepsilon(U)\,dx.$$

By defining

$$G_\varepsilon(t):=\int_0^t g_\varepsilon'(\tau)^{\frac{1}{p}}\,d\tau=t\,(\varepsilon+t)^{\frac{\alpha-1}{p}},$$

if we proceed as in the previous proof, we get

$$\begin{aligned}\left(\int_{\mathbb{R}^N} G_\varepsilon(U)^{p^*}\,dx\right)^{\frac{p}{p^*}} &\le \|U\|_\infty^{p^*+\alpha-1}\,|\{U>K_0\}|+\left(\int_{\{U\le K_0\}} U^{p^*}\,dx\right)^{\frac{p^*-p}{p^*}}\\ &\quad\times\left(\int_{\{U\le K_0\}}\Big(g_\varepsilon(U)\,U^{p-1}\Big)^{\frac{p^*}{p}}\,dx\right)^{\frac{p}{p^*}},\end{aligned}$$

for $K_0>0$. Observe that we also used the previous Lemma to assure that $U\in L^\infty(\mathbb{R}^N)$. We now observe that, by construction,

$$0\le g_\varepsilon(t)\le\int_0^t(\varepsilon+t)^{\alpha-1}\,d\tau=\frac{1}{\alpha}\,[(\varepsilon+t)^\alpha-\varepsilon^\alpha],$$

which implies

$$0\le g_\varepsilon(t)\,t^{p-1}\le\frac{1}{\alpha}\,[(\varepsilon+t)^\alpha-\varepsilon^\alpha]\,t^{p-1}\le\frac{1}{\alpha}\,(\varepsilon+t)^{\alpha-1}\,t^p=\frac{1}{\alpha}\,G_\varepsilon(t)^p.$$

Thus we arrive at

$$\left(\int_{\mathbb{R}^N} G_\varepsilon(U)^{p^*}\,dx\right)^{\frac{p}{p^*}}\le\|U\|_\infty^{p^*+\alpha-1}\,|\{U>K_0\}|+\frac{1}{\alpha}\left(\int_{\{U\le K_0\}} U^{p^*}\,dx\right)^{\frac{p^*-p}{p^*}}\left(\int_{\mathbb{R}^N} G_\varepsilon(U)^{p^*}\,dx\right)^{\frac{p}{p^*}},$$

The level $K_0=K_0(\alpha,U)>0$ is now chosen so that

$$\left(\int_{\{U\le K_0\}} U^{p^*}\,dx\right)^{\frac{p^*-p}{p^*}}\le\frac{\alpha}{2},$$

which yields

$$\left(\int_{\mathbb{R}^N}\Big(U\,(U+\varepsilon)^{\frac{\alpha-1}{p}}\Big)^{p^*}\,dx\right)^{\frac{p}{p^*}}\le 2\,\|U\|_\infty^{p^*+\alpha-1}\,|\{U>K_0\}|,$$

for every $0<\alpha<1$. By taking the limit as ε goes to 0, we get the desired integrability (3.4).

Part II: borderline Lorentz estimate. We now prove (3.5). For any $t>0$ we let $g_t(s)=\min\{t,\,s\}$, and define

$$G_t(s)=\int_0^s g_t'(\tau)^{\frac{1}{p}}\,d\tau=g_t(s).$$

We test equation (3.1) with $g_t(U)$ and, thanks to [5, Lemma A.2] and Sobolev inequality we get

$$\begin{split}\mathcal{S}_{p,s}\,\|g_t(U)\|_{p^*}^p\le[g_t(U)]_{s,p}^p&\le\int_{\mathbb{R}^{2N}}\frac{J_p(U(x)-U(y))\,\big(g_t(U(x))-g_t(U(y))\big)}{|x-y|^{N+s\,p}}\,dx\,dy\\&\le\mathcal{S}_{p,s}\int_{\mathbb{R}^N}U^{p^*-1}\,g_t(U)\,dx.\end{split}$$

We have $U\in L^{p^*-1}(\mathbb{R}^N)$, by choosing $q=p^*-1$ in (3.4). Thus we conclude that

$$t\,|\{U>t\}|^{\frac{1}{p^*}}\le\|g_t(U)\|_{p^*}\le\left(\int_{\mathbb{R}^N}U^{p^*-1}g_t(U)\,dx\right)^{\frac{1}{p}}\le t^{\frac{1}{p}}\,\|U\|_{p^*-1}^{\frac{p^*-1}{p}}.$$

This finally yields (3.5), after some elementary manipulations. □

3.2. **Decay estimates.** As an intermediate step towards the proof of the asymptotic result (1.9), in this subsection we will prove that any (positive) solution of (1.4) verifies

$$\frac{1}{C}\,|x|^{-\frac{N-s\,p}{p-1}}\le U(x)\le C\,|x|^{-\frac{N-s\,p}{p-1}},\qquad |x|>1,$$

for some $C=C(N,p,s,U)>1$, see Corollary 3.7 below.

In what follows, we will set for simplicity

$$\Gamma(x)=|x|^{-\frac{N-s\,p}{p-1}},\qquad x\in\mathbb{R}^N\setminus\{0\},$$

and

$$\widetilde{\Gamma}(x)=\min\{1,\,\Gamma(x)\}=\min\left\{1,\,|x|^{-\frac{N-s\,p}{p-1}}\right\},\qquad x\in\mathbb{R}^N. \tag{3.6}$$

The following expedient result will be useful.

Lemma 3.4. *With the notation above, we have*

$$\frac{1}{C}\,|x|^{-N-s\,p}\le(-\Delta_p)^s\widetilde{\Gamma}(x)\le C\,|x|^{-N-s\,p},\qquad \textit{for } |x|>R>1, \tag{3.7}$$

in weak sense, for some $C=C(N,p,s,R)>1$. The constant blows-up as R goes to 1.

Proof. From Theorem A.4, we know that Γ is a weak solution of $(-\Delta_p)^s u=0$ in B_R^c for any $R>1$. We then observe that the truncated function $\widetilde{\Gamma}$ can be written as

$$\widetilde{\Gamma}(x)=\Gamma(x)-\big(\Gamma(x)-1\big)_+.$$

Thus we apply Proposition 2.7, with the choices

$$\Omega=B_R^c,\qquad u=\Gamma,\qquad f\equiv 0,\qquad v=-(\Gamma-1)_+,$$

This yields for $|x| > R$

$$(-\Delta_p)^s\widetilde{\Gamma}(x) = 2\int_{B_1} \frac{J_p(\Gamma(x)-1) - J_p(\Gamma(x)-\Gamma(y))}{|x-y|^{N+s\,p}}\,dy$$
$$= 2\int_{B_1} \frac{J_p(\Gamma(y)-\Gamma(x)) - J_p(1-\Gamma(x))}{|x-y|^{N+s\,p}}\,dy. \tag{3.8}$$

We first prove the upper bound in (3.7). To this aim, by the monotonicity of Γ we get

$$(\Gamma(y)-\Gamma(x))^{p-1} - (1-\Gamma(x))^{p-1} \le (\Gamma(y)-\Gamma(x))^{p-1} \le \Gamma(y)^{p-1}, \qquad |x|>R,\ |y|\le 1.$$

Moreover

$$|x-y| \ge \frac{R-1}{R}\,|x|, \qquad \text{for all } |x|>R \text{ and } |y|<1.$$

By spending these informations in (3.8), we obtain

$$(-\Delta_p)^s\widetilde{\Gamma}(x) \le \left(\frac{R}{R-1}\right)^{N+s\,p} \frac{2}{|x|^{N+s\,p}} \int_{B_1} \Gamma(y)^{p-1}\,dy = \frac{C}{|x|^{N+s\,p}},$$

as desired. Observe that we also used that $\Gamma \in L^{p-1}_{\rm loc}(\mathbb{R}^N)$.

In order to prove the lower bound, we need to distinguish between the case $1<p<2$ and the case $p \ge 2$. If $p\ge 2$, then J_p is a convex superadditive function on $[0,\infty)$. Thus we get

$$J_p(\Gamma(y)-\Gamma(x)) - J_p(1-\Gamma(x)) \ge J_p(\Gamma(y)-1), \qquad |x|>R>1>|y|.$$

As for the kernel, we have

$$|x-y| < 2\,|x|, \qquad \text{if } |x|>|y|, \tag{3.9}$$

thus in conclusion from (3.8) we get

$$(-\Delta_p)^s\widetilde{\Gamma}(x) \ge \frac{2^{1-N-s\,p}}{|x|^{N+s\,p}} \int_{B_1} \big(\Gamma(y)-1\big)^{p-1}\,dy = \frac{C}{|x|^{N+s\,p}}.$$

By using again that $\Gamma \in L^{p-1}_{\rm loc}(\mathbb{R}^N)$ and that $\Gamma > 1$ in B_1, this gives the lower bound in (3.7), in the case $p\ge 2$.

In the case $1<p<2$, we need to use (2.3), which gives

$$J_p(\Gamma(y)-\Gamma(x)) - J_p(1-\Gamma(x)) \ge C\,\frac{(\Gamma(y)-1)}{\Big((\Gamma(y)-\Gamma(x))^2 + (1-\Gamma(x))^2\Big)^{\frac{2-p}{2}}}$$
$$\ge C\,\frac{(\Gamma(y)-1)}{(\Gamma(y)-\Gamma(x))^{2-p}} \ge C\,\Gamma(y)^{p-1}\left(1-\frac{1}{\Gamma(y)}\right).$$

By using this and (3.9) in (3.8), we get the desired lower bound for $1<p<2$ as well. □

In order to prove a lower bound for positive radially decreasing solutions of (1.4), we need to focus on the auxiliary problem

$$\mathcal{I}(R) = \inf_{u\in D^{s,p}(\mathbb{R}^N)} \Big\{[u]^p_{s,p} : u \ge \chi_{B_R}\Big\}. \tag{3.10}$$

Proposition 3.5. *Let $1 < p < \infty$ and $s \in (0,1)$ be such that $s\,p < N$. For any $R > 0$, problem (3.10) has a unique solution $u_R > 0$. Moreover, u_R is radial, non-increasing and $u_R \in D^{s,p}(\mathbb{R}^N)$ solves in weak sense*

$$\left\{\begin{array}{rcll}(-\Delta_p)^s u_R &=& 0, & \text{in } B_R^c,\\ u_R &\equiv& 1, & \text{in } B_R\end{array}\right.$$

Proof. Existence of a solution follows easily by using the Direct Methods. Indeed, if $\{u_n\}_{n\in\mathbb{N}} \subset D^{s,p}(\mathbb{R}^N)$ is a minimizing sequence, then a uniform bound on their Gagliardo seminorms entails a uniform bound on the L^{p^*} norms, by Sobolev inequality. Thus we have weak convergence (up to a subsequence) in $L^{p^*}(\mathbb{R}^N)$ to a function $u \in D^{s,p}(\mathbb{R}^N)$. Moreover, the convergence is strong in L^p on compact sets, as well. Thus the constraint $u_n \ge \chi_{B_R}$ passes to the limit and u is a minimizer. Uniqueness follows from strict convexity of the Gagliardo seminorm.

All the other required properties of u_R follows as in the proof of Proposition 3.1, we just show that u_R saturates the constraint $u_R \ge \chi_{B_R}$. For simplicity, we set $\mathcal{E}(u) = [u]^p_{s,p}$. Then from [15, Remark 3.3] we have

$$\mathcal{E}(\max\{u,\,t\}) + \mathcal{E}(\min\{u,\,t\}) \le \mathcal{E}(u), \qquad \text{for every } u \in D^{s,p}(\mathbb{R}^N),\ t \in \mathbb{R}. \tag{3.11}$$

In particular, $\min\{u_R, 1\}$ is still a minimizer and thus by uniqueness it coincides with u_R. $\square$

Thanks to Lemma 3.4, we can prove a decay estimate for the solution of (3.10).

Proposition 3.6. *The solution u_1 of problem (3.10) with $R = 1$ satisfies*

$$\frac{|x|^{-\frac{N-s\,p}{p-1}}}{C} \le u_1(x) \le C\,|x|^{-\frac{N-s\,p}{p-1}}, \qquad \text{for } |x| \ge 1,$$

for some constant $C = C(N,p,s) > 1$.

Proof. We prove the two estimates separately.

Upper bound. We first observe that by using the scaling properties of the Gagliardo seminorm, we have

$$\mathcal{I}(R) = R^{N-s\,p}\,\mathcal{I}(1). \tag{3.12}$$

For every $R > 1$, we set $u_1(R) = t \in (0,1)$. As in the previous proof, we set $\mathcal{E}(u) = [u]^p_{s,p}$. The function $\min\{u_1,\,t\}/t$ is admissible for problem (3.10) on B_R, then the minimality of u_R gives

$$\mathcal{E}\left(\frac{\min\{u_1,\,t\}}{t}\right) \ge \mathcal{E}(u_R) = \mathcal{I}(R) = R^{N-s\,p}\,\mathcal{I}(1),$$

thanks to (3.12). Similarly, we get

$$\mathcal{E}\left(\frac{\max\{u_1 - t,\,0\}}{1-t}\right) \ge \mathcal{E}(u_1) = \mathcal{I}(1).$$

then using the $p-$homogeneity of the energy and summing the previous two inequalities

$$\mathcal{E}(\min\{u_1,\,t\}) + \mathcal{E}(\max\{u_1,\,t\}) \ge \left(t^p\,R^{N-s\,p} + (1-t)^p\right)\mathcal{I}(1).$$

Using the submodularity of Gagliardo seminorms (3.11) in the left-hand side and simplifying we get

$$t^p\,R^{N-s\,p} \le 1 - (1-t)^p.$$

By recalling the definition of t, we obtain

$$u_1(R)^p\, R^{N-s\,p} \leq 1 - (1 - u_1(R))^p \tag{3.13}$$

and since $1 - (1 - u_1(R))^p \leq p\, u_1(R)$ we get

$$u_1(R) \leq p^{\frac{1}{p-1}}\, R^{-\frac{N-sp}{p-1}}.$$

Lower bound. By using Proposition 2.7 with

$$\Omega = B_3^c, \qquad u = u_1, \qquad f \equiv 0, \qquad v = -(u_1 - u_1(2))_+,$$

the truncated function

$$u = \min\{u_1,\, u_1(2)\} = u_1 - (u_1 - u_1(2))_+,$$

satisfies weakly in B_3^c

$$\begin{aligned}(-\Delta_p)^s u(x) &= 2 \int_{B_2} \frac{J_p(u_1(x) - u_1(2)) - J_p(u_1(x) - u_1(y))}{|x-y|^{N+s\,p}}\, dy \\ &\geq 2 \int_{B_1} \frac{J_p(u_1(y) - u_1(x)) - J_p(u_1(2) - u_1(x))}{|x-y|^{N+s\,p}}\, dy.\end{aligned}$$

In the last passage we used that the integrand is nonnegative by the monotonicity of u_1. Recall that $u_1 \equiv 1$ in B_1 and by (3.13) we have $u_1(2) < u_1(1)$. Then, it is readily checked that

$$(u_1(1) - u_1(x))^{p-1} - (u_1(2) - u_1(x))^{p-1} \geq c \qquad \forall |x| \geq 3,$$

for some constant $c = c(p, u_1(1) - u_1(2)) > 0$. Since also $|x - y| \leq 2\,|x|$ for all $x \in B_2^c$ and $y \in B_1$, the previous discussion yields

$$(-\Delta_p)^s u(x) \geq \frac{2\, c\, |B_1|}{(2\,|x|)^{N+s\,p}} =: \frac{c_1}{|x|^{N+s\,p}}, \qquad |x| \geq 3. \tag{3.14}$$

On the other hand, from Lemma 3.4, for every $\varepsilon > 0$ we have

$$(-\Delta_p)^s(\varepsilon\, \widetilde{\Gamma}(x)) \leq \frac{c_2}{|x|^{N+s\,p}}\, \varepsilon^{p-1}, \qquad x \in B_3^c. \tag{3.15}$$

The function $\widetilde{\Gamma}$ is the same defined in (3.6). Now choose $\varepsilon > 0$ as follows

$$\varepsilon = \min\left\{u_1(3),\, \left(\frac{c_1}{c_2}\right)^{\frac{1}{p-1}}\right\},$$

so that by (3.14) and (3.15) it holds

$$\left\{\begin{array}{rcll} (-\Delta_p)^s(\varepsilon\, \widetilde{\Gamma}) & \leq & (-\Delta_p)^s u, & \text{in } B_3^c, \\ \varepsilon\, \widetilde{\Gamma} & \leq & u, & \text{in } B_3. \end{array}\right.$$

Therefore by Theorem 2.6 and the definition of $\widetilde{\Gamma}$ we have

$$\varepsilon\, |x|^{-\frac{N-s\,p}{p-1}} \leq u = u_1, \qquad \text{in } B_3^c.$$

In $B_3 \setminus B_1$ the estimate is simpler to obtain, indeed

$$|x|^{-\frac{N-s\,p}{p-1}} \leq 1 \leq \frac{u_1(x)}{u_1(3)},$$

thus we get the conclusion. $\square$

Finally, we can prove the aforementioned decay estimate for solutions of (1.4).

Corollary 3.7 (Sharp decay rate)**.** *Let $U \in D^{s,p}(\mathbb{R}^N)$ be a positive radially symmetric and decreasing solution of* (1.4)*. Then*

$$C \left(\inf_{B_1} U\right) |x|^{-\frac{N-s\,p}{p-1}} \le U(x) \le \left(\omega_N^{-\frac{1}{p^*}} \|U\|_{\frac{p}{p^*-1}}^{\frac{p^*-1}{p}}\right)^{\frac{p}{p-1}} |x|^{-\frac{N-s\,p}{p-1}}, \qquad |x| \ge 1,$$

for some constant $C = C(N,p,s) > 0$.

Proof. The upper bound follows from the $L^{\vartheta,\infty}$ estimate of (3.5), combined with Lemma 2.8.

As for the lower bound, by the weak Harnack inequality for positive supersolution of $(-\Delta_p)^s$ (see [18, Theorem 5.2]), we have

$$\lambda := \inf_{B_1} U \ge C \left(\int_{B_2} U^{p-1}\,dx\right)^{\frac{1}{p-1}} > 0.$$

Then the function $\lambda\,u_1$ is a lower barrier for U. Thus the lower bound follows from Theorem 2.6 and Proposition 3.6. □

4. Proof of the main result

In this section we still denote by $\widetilde{\Gamma}$ the truncated function defined by (3.6). Since both U and $\widetilde{\Gamma}$ are radially symmetric, we will systematically use the abuse of notation $U(x) = U(r)$ and $\widetilde{\Gamma}(x) = \widetilde{\Gamma}(r)$, for $r = |x|$.

Lemma 4.1. *Suppose that*

$$U(R) \ge A\,\widetilde{\Gamma}(R), \qquad \text{for some } R > 2.$$

For any $\delta > 0$ there exists $\theta = \theta(N,p,s,\delta,U) < 1$ such that

$$U(r) \ge (A-\delta)\,\widetilde{\Gamma}(r), \qquad \text{for any } \theta\,R \le r \le R.$$

Similarly, if

$$U(R) \le B\,\widetilde{\Gamma}(R), \qquad \text{for some } R > 2,$$

then

$$U(r) \le (B+\delta)\,\widetilde{\Gamma}(r) \qquad \text{for any } R \le r \le R/\theta.$$

Proof. Consider the first statement and let $\theta < 1$ to be determined. U is non increasing and

$$U(r) \le C\,r^{-\frac{N-s\,p}{p-1}},$$

by Corollary 3.7. Then for any $\theta\,R \le r \le R$ it holds

$$\frac{U(R)}{\widetilde{\Gamma}(R)} - \frac{U(r)}{\widetilde{\Gamma}(r)} \le U(R)\left(\frac{1}{\widetilde{\Gamma}(R)} - \frac{1}{\widetilde{\Gamma}(r)}\right) \le \frac{C}{R^{\frac{N-s\,p}{p-1}}}\left(R^{\frac{N-s\,p}{p-1}} - r^{\frac{N-s\,p}{p-1}}\right) \le C\left(1-\theta^{\frac{N-s\,p}{p-1}}\right).$$

Therefore by hypothesis we get

$$\frac{U(r)}{\widetilde{\Gamma}(r)} \ge A - C\left(1-\theta^{\frac{N-s\,p}{p-1}}\right), \qquad \text{for } r \in [\theta\,R, R],$$

which gives the first claim. The proof of the other statement is similar: for any $R \le r \le R/\theta$ it holds

$$\frac{U(r)}{\widetilde{\Gamma}(r)} - \frac{U(R)}{\widetilde{\Gamma}(R)} \le U(R)\left(\frac{1}{\widetilde{\Gamma}(r)} - \frac{1}{\widetilde{\Gamma}(R)}\right) \le \frac{C}{R^{\frac{N-sp}{p-1}}}\left(r^{\frac{N-s\,p}{p-1}} - R^{\frac{N-s\,p}{p-1}}\right) \le C\left(\theta^{-\frac{N-s\,p}{p-1}} - 1\right),$$

which gives

$$\frac{U(r)}{\widetilde{\Gamma}(r)} \le B + C\left(\theta^{-\frac{N-sp}{p-1}} - 1\right), \qquad \text{for } r \in [R, R/\theta].$$

This completes the proof. □

We are ready for the proof of the main result.

Theorem 4.2. *There exists $C > 0$ such that*

$$\lim_{r\to+\infty} r^{\frac{N-s\,p}{p-1}}\, U(r) = C.$$

Proof. In the following, the dependence of the various constants from N, p and s will be omitted. Moreover, we can suppose that $p \ne 2$, since for $p = 2$ the function U has an explicit expression. By virtue of Corollary 3.7 we readily have

$$\frac{1}{C} \le m := \liminf_{r\to+\infty} \frac{U(r)}{\widetilde{\Gamma}(r)} \le \limsup_{r\to+\infty} \frac{U(r)}{\widetilde{\Gamma}(r)} =: M \le C,$$

with C depending on U as well. Suppose by contradiction that $M - m > 0$, and fix $0 < \varepsilon_0 < (M - m)/4$.

• <u>*Case $p > 2$*</u>. There exists $R_0 = R_0(\varepsilon_0) > 2$ such that

$$\frac{U(r)}{\widetilde{\Gamma}(r)} \ge m - \varepsilon_0, \qquad \text{for } r \ge R_0,$$

and we can choose an arbitrarily large $R > R_0$ such that

$$\frac{U(R)}{\widetilde{\Gamma}(R)} \ge M - \frac{M-m}{4}.$$

Consider $\delta = (M - m)/4$. By Lemma 4.1, there exists $\theta < 1$ so that for any such R it holds

$$\frac{U(r)}{\widetilde{\Gamma}(r)} \ge \frac{M+m}{2}, \qquad \text{for } r \in [\theta\, R, R]. \tag{4.1}$$

Since R can be chosen arbitrarily large, we can suppose $\theta R > R_0$. Consider, for any $0 < \varepsilon < (M - m)/4$, the lower barrier $w(r) = g(r)\,\widetilde{\Gamma}(r)$ where g is the following step function

$$g(r) = \begin{cases} 0, & \text{if } r < R_0 \\ m - \varepsilon_0, & \text{if } R_0 \le r < \theta R \\ \frac{M+m}{2}, & \text{if } \theta R \le r < \sqrt{\theta} R \\ m + \varepsilon, & \text{if } \sqrt{\theta}\, R < r \end{cases}$$

It is easily seen that $w \in \widetilde{D}^{s,p}(B_R^c)$. Moreover, by using (4.1), it is readily verified that $w \le U$ in B_R. We claim that, for sufficiently small ε_0 and ε and sufficiently large R, it holds

$$(-\Delta_p)^s w \le (-\Delta_p)^s U, \qquad \text{in } B_R^c.$$

This would end the proof, since Theorem 2.6 would yield $U \geq w$ in $\mathbb{R}^N$ and then

$$m = \liminf_{r\to+\infty} r^{\frac{N-s\,p}{p-1}}\, U(r) = \liminf_{r\to+\infty} \frac{U(r)}{\widetilde{\Gamma}(r)} \geq \liminf_{r\to+\infty} g(r) = m+\varepsilon,$$

giving a contradiction. The function $w-(m+\varepsilon)\,\widetilde{\Gamma}$ is supported in $B_{\sqrt{\theta}\,R} \Subset B_R$ and thus using Proposition 2.7 with

$$\Omega = B_R^c, \qquad u = (m+\varepsilon)\,\widetilde{\Gamma}, \qquad f = (-\Delta_p)^s\Big((m+\varepsilon)\,\widetilde{\Gamma}\Big), \qquad v = w-(m+\varepsilon)\,\widetilde{\Gamma},$$

and (3.7), for any $|x|>R$ it holds

$$\begin{aligned}(-\Delta_p)^s w(x) &= (m+\varepsilon)^{p-1}\,(-\Delta_p)^s\widetilde{\Gamma}(x)\\ &\quad+\int_{B_{\sqrt{\theta}\,R}} \frac{J_p\big((m+\varepsilon)\,\widetilde{\Gamma}(x)-w(y)\big)-J_p\big((m+\varepsilon)\,(\widetilde{\Gamma}(x)-\widetilde{\Gamma}(y))\big)}{|x-y|^{N+s\,p}}\,dy\\ &\leq \frac{C}{|x|^{N+s\,p}}+\int_{B_{\sqrt{\theta}\,R}} \frac{h(x,y)}{|x-y|^{N+s\,p}}\,dy,\end{aligned} \tag{4.2}$$

where

$$h(x,y) = J_p\big((m+\varepsilon)\,(\widetilde{\Gamma}(y)-\widetilde{\Gamma}(x))\big)-J_p\big(w(y)-(m+\varepsilon)\,\widetilde{\Gamma}(x)\big).$$

We now decompose the last integral in (4.2) as follows

$$\int_{B_{\sqrt{\theta}\,R}} dy = \int_{B_{R_0}} dy + \int_{B_{\theta\,R}\setminus B_{R_0}} dy + \int_{B_{\sqrt{\theta}\,R}\setminus B_{\theta\,R}} dy, \tag{4.3}$$

and proceed to estimate each term separately.

Being $R_0 = R_0(\varepsilon_0)$ and h universally bounded, it holds

$$\int_{B_{R_0}} \frac{h(x,y)}{|x-y|^{N+s\,p}}\,dy \leq \|h\|_{L^\infty(\mathbb{R}^N\times\mathbb{R}^N)}\,\frac{\omega_N\,R_0^N}{\Big||x|-R_0\Big|^{N+s\,p}} \leq \frac{C(\varepsilon_0)}{|x|^{N+s\,p}}\,(1-\theta)^{-N-s\,p}, \tag{4.4}$$

where we used that (recall that we are assuming $\theta\,R>R_0$)

$$\Big||x|-R_0\Big| \geq \left(1-\frac{R_0}{R}\right)\,|x| \geq (1-\theta)\,|x|, \qquad \text{for } x\in B_R^c.$$

For the second integral in (4.3), we notice that for $y\in B_{\theta\,R}\setminus B_{R_0}$ and $x\in B_R^c$ we have

$$h(x,y) = J_p\big((m+\varepsilon)\,(\widetilde{\Gamma}(y)-\widetilde{\Gamma}(x))\big)-J_p\big((m+\varepsilon)\,(\widetilde{\Gamma}(y)-\widetilde{\Gamma}(x))-(\varepsilon+\varepsilon_0)\,\widetilde{\Gamma}(y)\big)$$

Observe that by (2.1), with simple manipulations we get

$$h(x,y) \leq c\,\Big[(m+\varepsilon)^{p-2}+(\varepsilon+\varepsilon_0)^{p-2}\Big]\,(\varepsilon+\varepsilon_0)\,\widetilde{\Gamma}(y)^{p-1},$$

for $x\in B_R^c$, $y\in B_{\theta R}\setminus B_{R_0}$ and $c=c(p)>0$. Therefore, since

$$|x-y| \geq \Big||x|-|y|\Big| \geq |x|-\theta\,R \geq (1-\theta)\,|x|, \qquad \text{for } x\in B_R^c,\ y\in B_{\theta\,R},$$

recalling the definition of $\widetilde{\Gamma}$ we get

$$\begin{aligned}\int_{B_{\theta R}\setminus B_{R_0}} \frac{h(x,y)}{|x-y|^{N+s\,p}}\,dy &\le \frac{C\,(\varepsilon+\varepsilon_0)}{(1-\theta)^{N+s\,p}\,|x|^{N+s\,p}}\int_{B_{\theta R}\setminus B_{R_0}} \frac{1}{|y|^{N-s\,p}}\,dy \\ &\le C\,\frac{(\varepsilon+\varepsilon_0)}{(1-\theta)^{N+s\,p}}\,\frac{(\theta\,R)^{s\,p}}{|x|^{N+s\,p}},\end{aligned} \tag{4.5}$$

where $C = C(N,s,p,M+m) > 0$. For the third integral in (4.3), for $y \in B_{\sqrt{\theta}R}\setminus B_{\theta R}$ we have

$$\begin{aligned}h(x,y) &= J_p\Big((m+\varepsilon)(\widetilde{\Gamma}(y)-\widetilde{\Gamma}(x))\Big) - J_p\left((m+\varepsilon)\,(\widetilde{\Gamma}(y)-\widetilde{\Gamma}(x)) + \left(\frac{M-m}{2}-\varepsilon\right)\widetilde{\Gamma}(y)\right) \\ &\le J_p\Big(\underbrace{(m+\varepsilon)\,(\widetilde{\Gamma}(y)-\widetilde{\Gamma}(x))}_{a}\Big) - J_p\Big(\underbrace{(m+\varepsilon)\,(\widetilde{\Gamma}(y)-\widetilde{\Gamma}(x))}_{a} + \underbrace{\left(\frac{M-m}{4}\right)\widetilde{\Gamma}(y)}_{b}\Big),\end{aligned}$$

since $\varepsilon < (M-m)/4$. The inequality (2.2) thus gives

$$h(x,y) \le -2^{2-p}\left(\frac{M-m}{4}\right)^{p-1}\widetilde{\Gamma}(y)^{p-1}.$$

Therefore, using

$$|x-y| \le 2\,|x|, \qquad \text{for } x \in \mathbb{R}^N\setminus B_R,\ y \in B_{\sqrt{\theta}\,R},$$

we obtain

$$\begin{aligned}\int_{B_{\sqrt{\theta}\,R}\setminus B_{\theta\,R}} \frac{h(x,y)}{|x-y|^{N+s\,p}}\,dy &\le -\frac{c\,(M-m)^{p-1}}{|x|^{N+s\,p}}\int_{B_{\sqrt{\theta}\,R}\setminus B_{\theta\,R}} |y|^{s\,p-N}\,dy \\ &\le -c\,\theta^{\frac{s\,p}{2}}\left(1-\theta^{\frac{s\,p}{2}}\right)(M-m)^{p-1}\,\frac{R^{s\,p}}{|x|^{N+s\,p}},\end{aligned} \tag{4.6}$$

for a constant $c = c(N,s,p) > 0$. Gathering toghether the estimates (4.2), (4.4), (4.5) and (4.6) we proved

$$\begin{aligned}(-\Delta_p)^s w(x) &\le \left(C + \frac{C(\varepsilon_0)}{(1-\theta)^{N+s\,p}}\right)\frac{1}{|x|^{N+s\,p}} \\ &\quad - \left[c\left(1-\theta^{\frac{s\,p}{2}}\right)(M-m)^{p-1} - \frac{C\,(\varepsilon+\varepsilon_0)}{(1-\theta)^{N+s\,p}}\right]\frac{R^{s\,p}\,\theta^{s\,p}}{|x|^{N+s\,p}}.\end{aligned}$$

So we can choose $\varepsilon+\varepsilon_0$ small enough (depending only on N, p, s and $M-m$), so that the second term above is negative. Thus for any such a choice we have, for any $|x| \ge R$,

$$(-\Delta_p)^s w(x) \le \frac{C(\varepsilon_0)}{|x|^{N+s\,p}}, \qquad (-\Delta_p)^s U(x) = U(x)^{p^*-1} \ge \frac{1}{C\,|x|^{N+\frac{s\,p}{p-1}}},$$

where in the last estimate we used Corollary 3.7. Since $p > 2$, for sufficiently large R it holds

$$\frac{1}{C\,|x|^{N+\frac{s\,p}{p-1}}} \ge \frac{R^{s\,p\,\frac{p-2}{p-1}}}{C\,|x|^{N+s\,p}} \ge \frac{C\,(\varepsilon_0)}{|x|^{N+s\,p}},$$

and thus the claim follows.

- Case $1<p<2$. There exists $R_0=R_0(\varepsilon_0)>2$ such that

$$\frac{U(r)}{\widetilde{\Gamma}(r)}\le M+\varepsilon_0,\qquad \text{for } r\ge R_0$$

and we can choose an arbitrarily large $R>R_0$ such that

$$\frac{U(R)}{\widetilde{\Gamma}(R)}\le m+\frac{M-m}{4}.$$

As before, we consider $\delta=(M-m)/4$ in Lemma 4.1: there exists $\theta<1$ so that for any such R it holds

$$\frac{U(r)}{\widetilde{\Gamma}(r)}\le \frac{M+m}{2},\qquad \text{for every } r\in[R,R/\theta]. \tag{4.7}$$

Since $U\in L^\infty(\mathbb{R}^N)$, there exists $\overline{C}>0$ such that $U\le \overline{C}\,\widetilde{\Gamma}$ in $\mathbb{R}^N$, then for any $0<\varepsilon<(M-m)/4$ we consider the upper barrier $w(r)=g(r)\,\widetilde{\Gamma}(r)$, where

$$g(r)=\begin{cases}\overline{C}, & \text{if } r<R_0,\\ M+\varepsilon_0, & \text{if } R_0\le r<R,\\ \frac{M+m}{2}, & \text{if } R\le r<R/\sqrt{\theta},\\ M-\varepsilon, & \text{if } R/\sqrt{\theta}<r.\end{cases}$$

Again, it is easy to verify that $w\in \widetilde{D}^{s,p}(B_R^c)$. Using (4.7), we can verify that $w\ge U$ in $B_{R/\theta}$. We claim that, for sufficiently small ε_0 and ε and sufficiently large R, it holds

$$(-\Delta_p)^s w\ge (-\Delta_p)^s U,\qquad \text{in } B_{R/\theta}^c.$$

This would end the proof, since the comparison principle of Theorem 2.6 would yield $U\le w$ in $\mathbb{R}^N$ and then

$$M=\limsup_{r\to+\infty} r^{\frac{N-s\,p}{p-1}}\,U(r)=\limsup_{r\to+\infty}\frac{U(r)}{\widetilde{\Gamma}(r)}\le \limsup_{r\to+\infty} g(r)=M-\varepsilon,$$

which gives a contradiction. The function $w-(M-\varepsilon)\,\widetilde{\Gamma}$ is supported in $B_{R/\sqrt{\theta}}\Subset B_{R/\theta}$ and thus using again Proposition 2.7 with

$$\Omega=B_{R/\theta}^c,\qquad u=(M-\varepsilon)\,\widetilde{\Gamma},\qquad f=(-\Delta_p)^s\Big((M-\varepsilon)\,\widetilde{\Gamma}\Big),\qquad v=w-(M-\varepsilon)\,\widetilde{\Gamma},$$

and (3.7), for any $|x|>R/\theta$ it holds

$$\begin{aligned}(-\Delta_p)^s w(x)&=(M-\varepsilon)^{p-1}\,(-\Delta_p)^s\,\widetilde{\Gamma}(x)\\ &\quad+\int_{B_{R/\sqrt{\theta}}}\frac{J_p\big((M-\varepsilon)\,\widetilde{\Gamma}(x)-w(y)\big)-J_p\big((M-\varepsilon)\,(\widetilde{\Gamma}(x)-\widetilde{\Gamma}(y))\big)}{|x-y|^{N+s\,p}}\,dy\\ &\ge \frac{1}{C\,|x|^{N+sp}}+\int_{B_{R/\sqrt{\theta}}}\frac{h(x,y)}{|x-y|^{N+sp}}\,dy,\end{aligned} \tag{4.8}$$

where

$$h(x,y)=J_p\big((M-\varepsilon)\,(\widetilde{\Gamma}(y)-\widetilde{\Gamma}(x))\big)-J_p\big(w(y)-(M-\varepsilon)\,\widetilde{\Gamma}(x)\big).$$

As above, we now decompose the last integral in (4.8)

$$\int_{B_{R/\sqrt{\theta}}} dy = \int_{B_{R_0}} dy + \int_{B_R\setminus B_{R_0}} dy + \int_{B_{R/\sqrt{\theta}}\setminus B_R} dy,$$

and proceed to estimate each term separately.

Being $R_0 = R_0(\varepsilon_0)$ and h universally bounded, as before we get

$$\int_{B_{R_0}} \frac{h(x,y)}{|x-y|^{N+sp}}\, dy \geq -\frac{C(\varepsilon_0)}{|x|^{N+s\,p}}, \tag{4.9}$$

where this time we used that (recall that we are assuming $R > R_0$)

$$\Big||x| - R_0\Big| \geq \left(1 - \frac{R_0}{R}\,\theta\right)\,|x| \geq (1-\theta)\,|x|, \qquad \text{for } x \in B^c_{R/\theta}.$$

For $y \in B_R \setminus B_{R_0}$ we have

$$h(x,y) = J_p\Big((M-\varepsilon)\,(\widetilde{\Gamma}(y) - \widetilde{\Gamma}(x))\Big) - J_p\Big((M-\varepsilon)\,(\widetilde{\Gamma}(y) - \widetilde{\Gamma}(x)) + (\varepsilon+\varepsilon_0)\,\widetilde{\Gamma}(y)\Big),$$

and by subadditivity of $\tau \mapsto \tau^{p-1}$, we get

$$h(x,y) \geq -(\varepsilon+\varepsilon_0)^{p-1}\,\widetilde{\Gamma}(y)^{p-1}.$$

Therefore, the analogous of (4.5) is now

$$\int_{B_R\setminus B_{R_0}} \frac{h(x,y)}{|x-y|^{N+sp}}\, dy \geq -C\,(\varepsilon+\varepsilon_0)^{p-1}\,\frac{R^{s\,p}}{|x|^{N+s\,p}}, \tag{4.10}$$

and again $C = C(N,s,p,M+m) > 0$. For the previous estimate we also used that

$$|x-y| \geq \Big||x| - |y|\Big| \geq |x| - R \geq (1-\theta)\;|x|, \qquad \text{for } x \in B^c_{R/\theta},\ y \in B_R.$$

For $y \in B_{R/\sqrt{\theta}} \setminus B_R$ and $x \in B^c_{R/\theta}$ we have

$$h(x,y) = J_p\Big(\underbrace{(M-\varepsilon)\,(\widetilde{\Gamma}(y) - \widetilde{\Gamma}(x))}_{a}\Big) - J_p\Big(\underbrace{(M-\varepsilon)\,(\widetilde{\Gamma}(y) - \widetilde{\Gamma}(x))}_{a} - \underbrace{\left(\frac{M-m}{2} - \varepsilon\right)\,\widetilde{\Gamma}(y)}_{b}\Big).$$

Clearly

$$0 \leq a = (M-\varepsilon)\,(\widetilde{\Gamma}(y) - \widetilde{\Gamma}(x)) \leq (M-\varepsilon)\,\widetilde{\Gamma}(y) =: A,$$

so that (2.4) provides

$$h(x,y) \geq \max\left\{(M-\varepsilon)^{p-1} - \left(\frac{M+m}{2}\right)^{p-1},\, \left(\frac{M-m}{2} - \varepsilon\right)^{p-1} 2^{1-p}\right\}\,\widetilde{\Gamma}(y)^{p-1}.$$

Proceeding as for (4.6) and using

$$|x-y| \leq 2\,|x|, \qquad \text{for } x \in B^c_{R/\theta},\ y \in B_{R/\sqrt{\theta}},$$

we thus obtain

$$\int_{B_{R/\sqrt{\theta}}\setminus B_R} \frac{h(x,y)}{|x-y|^{N+s\,p}}\, dy \geq \frac{c}{|x|^{N+s\,p}} \int_{B_{R/\sqrt{\theta}}\setminus B_R} |y|^{s\,p-N}\, dy \geq c\,\frac{R^{s\,p}}{|x|^{N+s\,p}}, \tag{4.11}$$

for a small constant c depending only on M and m. Gathering together the estimates (4.8), (4.9), (4.10) and (4.11), we proved

$$(-\Delta_p)^s w(x) \geq -\frac{C(\varepsilon_0)}{|x|^{N+s\,p}} + \left(c - C\,(\varepsilon+\varepsilon_0)^{p-1}\right)\frac{R^{s\,p}}{|x|^{N+s\,p}}.$$

in $B^c_{R/\theta}$. We can thus choose ε_0 and ε small enough so that the second term above is positive. For any such choice we have, for any $|x| \geq R/\theta$,

$$(-\Delta_p)^s w(x) \geq -\frac{C(\varepsilon_0)}{|x|^{N+s\,p}} + \frac{c}{2}\frac{R^{s\,p}}{|x|^{N+s\,p}},$$

and for sufficiently large R so that $c\,R^{s\,p} > 4\,C(\varepsilon_0)$ it holds

$$(-\Delta_p)^s w(x) \geq \frac{c}{4}\frac{R^{s\,p}}{|x|^{N+s\,p}}.$$

By using Corollary 3.7 and the fact that $1 < p < 2$, for every $|x| \geq R/\theta$ we get

$$(-\Delta_p)^s U(x) = U^{p^*-1}(x) \leq \frac{C}{|x|^{N+\frac{s\,p}{p-1}}} \leq \frac{C\,\theta^{s\,p\,\frac{2-p}{p-1}}}{R^{s\,p\,\frac{2-p}{p-1}}\,|x|^{N+s\,p}}.$$

We thus conclude that $(-\Delta_p)^s U \leq (-\Delta_p)^s w$ in $B^c_{R/\theta}$ for R sufficiently large. □

Appendix A. Power functions

We have the following result on power functions.

Lemma A.1. *Let $0 < (N-s\,p)/p < \beta < N/(p-1)$. For every $R > 0$, the function $x \mapsto |x|^{-\beta}$ belongs to $\widetilde{D}^{s,p}(B^c_R)$.*

Proof. A direct computation shows that $x \mapsto |x|^{-\beta}$ belongs to $L^{p-1}_{\rm loc}(\mathbb{R}^N) \cap L^{p^*}(B^c_R)$, when β is as in the statement. To show that

$$\left[|x|^{-\beta}\right]_{W^{s,p}(B^c_R)} < +\infty, \qquad \text{for } \frac{N-s\,p}{p} < \beta, \tag{A.1}$$

we compute

$$\begin{aligned}\int_{B^c_R\times B^c_R} \frac{||x|^{-\beta}-|y|^{-\beta}|^p}{|x-y|^{N+sp}}\,dx\,dy &= \int_{\mathbf{S}^{N-1}\times\mathbf{S}^{N-1}}\int_R^{+\infty}\int_R^{+\infty} \frac{|\varrho^{-\beta}-t^{-\beta}|^p\,\varrho^{N-1}\,t^{N-1}}{|\varrho\,\omega_1 - t\,\omega_2|^{N+sp}}\,d\varrho\,dt\,d\omega_1\,d\omega_2\\ &= 2\int_R^{+\infty}\frac{\varrho^{-\beta\,p}\,\varrho^{2\,N-2}}{\varrho^{N+sp}}\int_R^{\varrho}\left|1-\left(\frac{t}{\varrho}\right)^{-\beta}\right|^p\int_{\mathbf{S}^{N-1}\times\mathbf{S}^{N-1}}\frac{d\omega_1\,d\omega_2}{|\omega_1-(t\,\omega_2)/\varrho|^{N+s\,p}}\left(\frac{t}{\varrho}\right)^{N-1}dt\,d\varrho\\ &= 2\int_R^{+\infty}\frac{\varrho^{-\beta\,p}\,\varrho^{2\,N-1}}{\varrho^{N+s\,p}}\int_{R/\varrho}^1 |1-\xi^{-\beta}|^p\,\xi^{N-1}\int_{\mathbf{S}^{N-1}\times\mathbf{S}^{N-1}}\frac{d\omega_1\,d\omega_2}{|\omega_1-\xi\,\omega_2|^{N+s\,p}}\,d\xi\,d\varrho.\end{aligned}$$

Let us now prove that for $0 < \xi < 1$ it holds

$$\int_{\mathbf{S}^{N-1}\times\mathbf{S}^{N-1}}\frac{d\omega_1\,d\omega_2}{|\omega_1-\xi\,\omega_2|^{N+s\,p}} \leq \frac{C}{(1-\xi)^{1+s\,p}}.$$

Without loss of generality, we may assume that $\xi \geq 1/2$, since for $0<\xi<1/2$ the integral is uniformly bounded. By rotational invariance, we have

$$\int_{\mathbf{S}^{N-1}\times\mathbf{S}^{N-1}} \frac{d\omega_1\, d\omega_2}{|\omega_1-\xi\omega_2|^{N+sp}} = |\mathbf{S}^{N-1}| \int_{\mathbf{S}^{N-1}} \frac{d\omega_2}{|\mathbf{e}_1-\xi\omega_2|^{N+sp}},$$

where $\mathbf{e}_1=(1,0,\dots,0)$. By changing variable $\omega_2=(t,z)$ with

$$t=\pm\sqrt{1-|z|^2}, \qquad z\in B'_1\subset\mathbb{R}^{N-1},$$

we therefore get

$$\begin{aligned}\int_{\mathbf{S}^{N-1}} \frac{d\omega_2}{|\mathbf{e}_1-\xi\,\omega_2|^{N+s\,p}} &= \int_{\mathbf{S}^{N-1}\setminus B_1(\mathbf{e_1})} \frac{d\omega_2}{|\mathbf{e}_1-\xi\,\omega_2|^{N+s\,p}} + \int_{\mathbf{S}^{N-1}\cap B_1(\mathbf{e_1})} \frac{d\omega_2}{|\mathbf{e}_1-\xi\,\omega_2|^{N+s\,p}}\\
&\leq C\left(1+\int_{B'_1}\frac{dz}{((1-\xi\,t)^2+\xi^2\,|z|^2)^{\frac{N+s\,p}{2}}}\right)\\
&\leq C\left(1+\int_{B'_1}\frac{dz}{((1-\xi)^2+\xi^2\,|z|^2)^{\frac{N+s\,p}{2}}}\right)\\
&\leq C\left(1+\frac{1}{(1-\xi)^{1+s\,p}}\int_{B'_{\frac{\xi}{1-\xi}}}\frac{1}{(1+|y|^2)^{\frac{N+s\,p}{2}}}dy\right)\\
&\leq C\left(1+\frac{1}{(1-\xi)^{1+s\,p}}\int_{\mathbb{R}^{N-1}}\frac{1}{(1+|y|^2)^{\frac{N+s\,p}{2}}}dy\right)\end{aligned}$$

which proves the claim. Taking into account that for $0<\xi<1$ it also holds

$$\frac{|1-\xi^{-\beta}|^p}{|1-\xi|^{1+s\,p}} \leq C\,(\xi^{-\beta\,p}+|1-\xi|^{p\,(1-s)-1})$$

we therefore get

$$\Big[|x|^{-\beta}\Big]^p_{W^{s,p}(B_R^c)} \leq C\int_R^{+\infty} \varrho^{N-1-p\,(s+\beta)}\,d\varrho \int_{R/\varrho}^1 \xi^{N-1}\left(\xi^{-\beta\,p}+|1-\xi|^{p\,(1-s)-1}\right)d\xi.$$

All the integrals are now explicitly computable and one can readily get (A.1). □

Lemma A.2. *Let $0<(N-s\,p)/p<\beta<N/(p-1)$. For every $R>0$, it holds*

$$(-\Delta_p)^s|x|^{-\beta} = C(\beta)\,|x|^{-\beta\,(p-1)-s\,p} \qquad \textit{weakly in } B_R^c,$$

where the constant $C(\beta)$ is given by

$$C(\beta) = 2\int_0^1 \varrho^{s\,p-1}\left[1-\varrho^{N-s\,p-\beta\,(p-1)}\right]\left|1-\varrho^\beta\right|^{p-1}\Phi(\varrho)\,d\varrho, \tag{A.2}$$

and

$$\Phi(\varrho) = \mathcal{H}^{N-2}(\mathbf{S}^{N-2})\int_{-1}^1 \frac{(1-t^2)^{\frac{N-3}{2}}}{\left(1-2\,t\,\varrho+\varrho^2\right)^{\frac{N+s\,p}{2}}}\,dt. \tag{A.3}$$

Proof. Observe that

$$|x|^{-\beta\,(p-1)-s\,p}\in L^{(p*)'}(B_R^c),\qquad \text{for any } \beta>(N-s\,p)/p.$$

Then, by Theorem 2.1 and Proposition 2.4 it suffices to test the weak form of the equation with an arbitrary $\varphi\in C^\infty_c(B_R^c)$. For every such a φ we consider the double integral

$$\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{J_p(|x|^{-\beta}-|y|^{-\beta})}{|x-y|^{N+s\,p}}\,(\varphi(x)-\varphi(y))\,dx\,dy.$$

We observe that the integral is absolutely convergent, indeed

$$\begin{aligned}\int_{\mathbb{R}^N\times\mathbb{R}^N}&\frac{|J_p(|x|^{-\beta}-|y|^{-\beta})|}{|x-y|^{N+s\,p}}\,|\varphi(x)-\varphi(y)|\,dx\,dy\\ &=\int_{B_R^c\times B_R^c}\frac{|J_p(|x|^{-\beta}-|y|^{-\beta})|}{|x-y|^{N+s\,p}}\,|\varphi(x)-\varphi(y)|\,dx\,dy\\ &+2\int_{B_R}\int_{\mathrm{supp}(\varphi)}\frac{|J_p(|x|^{-\beta}-|y|^{-\beta})|}{|x-y|^{N+s\,p}}\,|\varphi(y)|\,dx\,dy\\ &\le\Big[|x|^{-\beta}\Big]_{W^{s,p}(B_R^c)}\,[\varphi]_{W^{s,p}(B_R^c)}+C\,\|\varphi\|_{L^\infty}\,|\mathrm{supp}(\varphi)|\int_{B_R}|x|^{-\beta\,(p-1)}\,dx,\end{aligned}$$

and both terms are finite, thanks to Lemma A.1. For $\delta>0$ we consider the conical set

$$\mathcal{O}_\delta=\{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N\,:\,(1-\delta)\,|x|\le|y|\le(1+\delta)\,|x|\},$$

then by the Dominated Convergence Theorem

$$\begin{aligned}\lim_{\delta\searrow 0}\int_{\mathcal{O}_\delta^c}&\frac{J_p(|x|^{-\beta}-|y|^{-\beta})}{|x-y|^{N+s\,p}}\,(\varphi(x)-\varphi(y))\,dy\,dx\\ &=\int_{\mathbb{R}^{2N}}\frac{J_p(|x|^{-\beta}-|y|^{-\beta})}{|x-y|^{N+s\,p}}\,(\varphi(x)-\varphi(y))\,dx\,dy.\end{aligned}$$

We now observe that

$$\int_{\mathcal{O}_\delta^c}\frac{J_p(|x|^{-\beta}-|y|^{-\beta})}{|x-y|^{N+s\,p}}\,(\varphi(x)-\varphi(y))\,dy\,dx=2\int_{\mathbb{R}^N}\left(\int_{\mathcal{K}_\delta(x)^c}\frac{J_p(|x|^{-\beta}-|y|^{-\beta})}{|x-y|^{N+s\,p}}\,dy\right)\varphi(x)\,dx,$$

where for every $x\in\mathbb{R}^N$

$$\mathcal{K}_\delta(x)=\{y\in\mathbb{R}^N\,:\,(1-\delta)\,|x|\le|y|\le(1+\delta)\,|x|\},$$

and of course $\mathcal{K}_\delta(x)=\mathcal{K}_\delta(x')$ whenever $|x|=|x'|$. We set

$$f_\delta(x)=2\int_{\mathcal{K}_\delta(x)^c}\frac{J_p(|x|^{-\beta}-|y|^{-\beta})}{|x-y|^{N+s\,p}}\,dy,\qquad x\in\mathbb{R}^N\setminus\{0\},$$

it is easily seen that f_δ is a radial function, homogeneous of degree $-\beta\,(p-1)-s\,p$ (see [4, Lemma 6.2]). Thus for $x\neq 0$ we have

$$f_\delta(x)=|x|^{-\beta\,(p-1)-s\,p}\,f_\delta(\omega),\qquad\text{for}\quad\omega=\frac{x}{|x|}\in\mathbf{S}^{N-1}.\tag{A.4}$$

We set

$$C(\beta;\delta):=f_\delta(\omega)=2\int_{\mathcal{K}_\delta(\omega)^c}\frac{J_p(1-|y|^{-\beta})}{|\omega-y|^{N+s\,p}}\,dy,\qquad\omega\in\mathbf{S}^{N-1},$$

which is independent of the direction ω, by radiality of f_δ. By taking the average over $\mathbf{S}^{N-1}$ and proceeding as in [4, Lemma B.2], we get

$$C(\beta;\delta) = 2\int_{|\varrho-1|\geq\delta} \varrho^{N-1}\,|1-\varrho^{-\beta}|^{p-2}\,(1-\varrho^{-\beta})\,\Phi(\varrho)\,d\varrho,$$

where Φ is defined in (A.3). We now decompose the integral defining $C(\beta;\delta)$ and perform a change of variables, i.e.

$$\begin{aligned}
C(\beta;\delta) &= -2\int_0^{1-\delta} \varrho^{N-1}\,|1-\varrho^{-\beta}|^{p-1}\,\Phi(\varrho)\,d\varrho \\
&\quad + 2\int_{1+\delta}^{\infty} \varrho^{N-1}\,|1-\varrho^{-\beta}|^{p-1}\,\Phi(\varrho)\,d\varrho \\
&= -2\int_0^{1-\delta} \varrho^{N-1-\beta\,(p-1)}\,|\varrho^{\beta}-1|^{p-1}\,\Phi(\varrho)\,d\varrho \\
&\quad + 2\int_0^{1/(1+\delta)} \varrho^{-N-1}\,|1-\varrho^{\beta}|^{p-1}\,\Phi(1/\varrho)\,d\varrho.
\end{aligned}$$

Finally, observe that

$$\Phi(1/\varrho) = \varrho^{N+s\,p}\,\Phi(\varrho),$$

thus the quantity $C(\beta;\delta)$ can be written as

$$\begin{aligned}
C(\beta;\delta) &= 2\int_0^{1-\delta} \left(1-\varrho^{N-s\,p-\beta\,(p-1)}\right)\,\varrho^{s\,p-1}\,(1-\varrho^{\beta})^{p-1}\,\Phi(\varrho)\,d\varrho \\
&\quad + 2\int_{1-\delta}^{1/(1+\delta)} \varrho^{s\,p-1}\,(1-\varrho^{\beta})^{p-1}\,\Phi(\varrho)\,d\varrho.
\end{aligned}$$

Recall that φ is supported in B_R^c, thus by using (A.4) we can estimate

$$\left|\int_\Omega f_\delta\,\varphi\,dx - C(\beta)\int_\Omega |x|^{-\beta\,(p-1)-s\,p}\,\varphi\,dx\right| \leq \|\varphi\|_\infty\,R^{-\beta\,(p-1)-s\,p}\,|\mathrm{supp}(\varphi)|\,\Big|C(\beta;\delta)-C(\beta)\Big|.$$

In order to that $C(\beta;\delta)$ converges to $C(\beta)$ as δ goes to 0, we decompose the function Φ defined in (A.3) as follows

$$\begin{aligned}
\Phi(\varrho) &= \int_{-1}^{1/2} \frac{(1-t^2)^{\frac{N-3}{2}}}{(1-2\,t\,\varrho+\varrho^2)^{\frac{N+s\,p}{2}}}\,dt + \int_{1/2}^{1} \frac{(1-t^2)^{\frac{N-3}{2}}}{(1-2\,t\,\varrho+\varrho^2)^{\frac{N+s\,p}{2}}}\,dt \\
&=: \Phi_1(\varrho)+\Phi_2(\varrho),
\end{aligned}$$

where we omitted the dimensional constant $\mathcal{H}^{N-1}(\mathbf{S}^{N-2})$ for simplicity. If we use that

$$1-2\,t\,\varrho+\varrho^2 = (\varrho-t)^2+(1-t^2) \geq \frac{3}{4}, \qquad \text{if } -1\leq t\leq \frac{1}{2},$$

we get

$$\Phi_1(\varrho)\leq C, \qquad 0<\varrho<1. \tag{A.5}$$

We now consider $\Phi_2(\varrho)$, here we discuss separately the cases $0<\varrho<1/2$ and $1/2\leq\varrho<1$. We observe that for $0<\varrho<1/2$ we have

$$1-2\,t\,\varrho+\varrho^2 = (1-\varrho)^2+2\,\varrho\,(1-t) \geq \frac{1}{4}, \qquad \text{if } \frac{1}{2}\leq t\leq 1.$$

Then we get again

$$\Phi_2(\varrho) \le C, \qquad 0 < \varrho < \frac{1}{2}. \tag{A.6}$$

We are left with the term Φ_2 for $1/2 \le \varrho < 1$. With simple manipulations[3] we can write it as

$$\Phi_2(\varrho) = \frac{(2\,\varrho)^{-\frac{N-1}{2}}}{(1-\varrho)^{1+s\,p}} \int_0^{\frac{\varrho}{(1-\varrho)^2}} \frac{\left(2 - \frac{(1-\varrho)^2}{2\,\varrho}\,\tau\right)^{\frac{N-3}{2}}\,\tau^{\frac{N-3}{2}}}{(1+\tau)^{\frac{N+s\,p}{2}}}\,d\tau.$$

In particular, we get

$$\Phi_2(\varrho) \le C\,(1-\varrho)^{-1-s\,p}, \qquad 1 > \varrho \ge \frac{1}{2}. \tag{A.7}$$

By using (A.5), (A.6) and (A.7), we thus obtain

$$\lim_{\delta\searrow 0} 2\int_0^{1-\delta} \left(1-\varrho^{N-s\,p-\beta\,(p-1)}\right)\,\varrho^{s\,p-1}\,(1-\varrho^\beta)^{p-1}\,\Phi(\varrho)\,d\varrho = C(\beta),$$

and observe that the latter is finite, thanks to (A.7). For the other integral, still by (A.5) and (A.7), we obtain

$$\begin{aligned}\lim_{\delta\searrow 0}\int_{1-\delta}^{1/(1+\delta)} \varrho^{s\,p-1}\,(1-\varrho^\beta)^{p-1}\,\Phi(\varrho)\,d\varrho &\le C\,\lim_{\delta\searrow 0}\int_{1-\delta}^{1/(1+\delta)} \varrho^{s\,p-1}\,(1-\varrho^\beta)^{p-1}\,d\varrho\\ &\quad + C\,\lim_{\delta\searrow 0}\int_{1-\delta}^{1/(1+\delta)} \varrho^{s\,p-1}\,(1-\varrho^\beta)^{p-1}\,(1-\varrho)^{-1-s\,p}\,d\varrho\\ &\le C\,\lim_{\delta\searrow 0}\int_{1-\delta}^{1/(1+\delta)} (1-\varrho)^{p-2-s\,p}\,d\varrho\\ &\le \frac{C}{p-1-s\,p}\,\lim_{\delta\searrow 0}\left[\left(\frac{\delta}{1+\delta}\right)^{p-1-s\,p} - \delta^{p-1-s\,p}\right],\end{aligned}$$

where we assumed for simplicity that $p - s - s\,p \ne 0$. If $p - 1 - s\,p > 0$, the last term converges to 0. If $p - 1 - s\,p < 0$, we have

$$\left(\frac{\delta}{1+\delta}\right)^{p-1-s\,p} - \delta^{p-1-s\,p} \simeq \delta^{p-1-s\,p}\left[(1+\delta)^{s\,p+1-p} - 1\right] \simeq \delta^{p-s\,p}, \qquad \text{as } \delta\searrow 0,$$

and thus the integral converges to 0 again. Finally, the case $p-1-s\,p = 0$ is treated similarly, we leave the details to the reader.

In conclusion, we get

$$\lim_{\delta\searrow 0}\int_\Omega f_\delta\,\varphi\,dx = C(\beta)\int_\Omega |x|^{-\beta\,(p-1)-s\,p}\,\varphi\,dx,$$

as desired. □

[3] We use the change of variables

$$\tau = \frac{2\,\varrho}{(1-\varrho)^2}\,(1-t).$$

Remark A.3. The previous result was proved in [14, Lemma 3.1] for the limit case $\beta = (N - s\,p)/p$. Our argument is different, since we rely on elementary estimates for the function Φ, rather than on special properties of hypergeometric and beta functions like in [14].

Observe that the choice $\beta = (N - s\,p)/(p-1)$ is feasible in the previous results, since

$$\frac{N - s\,p}{p} < \frac{N - s\,p}{p-1} < \frac{N}{p-1}.$$

Moreover, with such a choice we have $C(\beta) = 0$ in (A.2). Then from Lemmas A.1 and A.2, we get the following.

Theorem A.4. *For any $R > 0$, $\Gamma(x) = |x|^{-\frac{N-s\,p}{p-1}}$ belongs to $\widetilde{D}^{s,p}(B_R^c)$ and weakly solves $(-\Delta_p)^s u = 0$ in B_R^c.*

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